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THE FIFTY-SECOND REGULAR MEETING OF THE SAN FRANCISCO SECTION

The fifty-second regular meeting of the San Francisco Section of the Society was held at the University of British Columbia on Saturday, June 18, 1927. The total attendance was thirty, including the following fifteen members of the Society:

Bernstein, Biggerstaff, Blichfeldt, Daniel Buchanan, J. W. Campbell, Carpenter, DeCou, Griffin, McAlister, W. E. Milne, Moritz, Mullemeister, Neikirk, F. S. Nowlan, Winger.

Professor Daniel Buchanan was elected temporary chairman. President Klinck of the University of British Columbia welcomed the visiting members. The Section accepted the cordial invitation of Professor Griffin to hold the next Summer meeting at Reed College. After the meeting the visitors were entertained at a luncheon on the Campus followed by a drive through the city of Vancouver.

Titles and abstracts of papers read at the meeting follow. The papers of Professors Bell, Buchanan, Davis, Dines, Moore, Dr. Trjitzinsky, Mr. Gage, Mr. Jerbert, and Dr. Whyburn were read by title. The papers of Miss Deutsch, Miss Haack, Miss Pennock, Miss Rollins, and Mr. So were sponsored and read by Professor Griffin. Mr. Gage's paper was sponsored by Professor Buchanan.

1. Professor E. T. Bell: *Cauchy's cyclotomic function and functional powers.*

This paper has appeared in the July-August number of this Bulletin.

2. Professor E. T. Bell: *Numerical functions of multipartite integers and compound partitions.*

An arithmetic of hypercomplex numbers with r positive integral coefficients is developed. The divisibility properties of functions of such numbers are a logical resultant of those of Dirichlet multiplication and the process of elliptic multiplication and the processes of elliptic functions; they give rise to new aspects of compound partitions.

3. Professor E. T. Bell: *Partition polynomials.*

A class of polynomials depending on all the partitions of a given integer into unequal parts is defined. For special values of the s independent variables in the nth polynomial, the value is the number of representations of n in the form $cx \cdots xx$ where $c, \cdots$ are integers 0; for other special values the nth polynomial is the binary quadratic class number for the negative determinant n.

4. Professor Daniel Buchanan: *The Trojan asymptotic satellites.*

An earlier paper discussed the asymptotic satellites in the vicinity of the equilateral triangle equilibrium points, but the analysis was not valid for certain small ratios of the finite masses, for example, that of Jupiter and the sun. A different treatment of the problem is now given which is valid for certain small ratios only.

5. Professor A. F. Carpenter: *Complete systems of invariants and covariants of a certain system of surfaces.*

An analytic basis for the projective differential geometry of triads of ruled surfaces whose generators are in one-to-one correspondence has been established in a paper soon to appear in the Transactions. As a sequel the present paper develops with reasonable completeness the theory of the invariants and covariants of such a system of surfaces.

6. Professor D. R. Davis: *Characteristic properties of differential equations of the calculus of variations.*

Self-adjoint systems of differential equations of the second order are discussed. For a given system of differential equations of the second order the unique existence of an adjoint system is proved. Necessary and sufficient conditions for a given system of the second order to be self-adjoint are established. Characteristic properties of differential equations of the calculus of variations are then derived. It is here shown that if two equations of the form $H(x, y, z, y', z', y'', z'')=0$, $K(x, y, z, y', z', y'', z'')=0$ are to be the Euler equations for the integral $I=\int_{x_1}^{x_2} f(x, y, z, y', z')dx$, they must have their equations of variation $H_y u+H_{y'}u'+H_{z'}v'+H_{y''}u''+H_{z''}v''=0$, $K_y u+K_{z'}v+K_{y'}u'+K_{z'}v'+K_{y''}u''+K_{z''}v''=0$, self-adjoint.

7. Professor D. R. Davis: *Integrals whose extremals have given differential equations.*

That the equations of variation of a given system of differential equations of the second order, $H=K=0$, must be self-adjoint in order that the given equations be the Euler equations for an integral I was established as a necessary condition in the preceding paper. The sufficiency of this property is proved here. Necessary and sufficient conditions for the system $H=K=0$ to be self-adjoint are given. Also the most general form of the

integrand f for an integral I whose Euler equations are the equations $H=K=0$ is determined.

8. Professor D. R. Davis: *A method of determining an integral I whose extremals are a given family of curves.*

The problem of finding an integral whose extremals are a given family of curves is here treated. Necessary and sufficient conditions for the two given equations of the form $y''=F(x, y, z, y', z')$, $z''=G(x, y, z, y', z')$ to have solutions $y=y(x)$, $z=z(x)$ which represent the extremals for an integral I, of the form expressed in abstract 6, are derived. Also the possibility of determining linear conbinations of the two given equations $y''=F$, $z''=G$, which shall have equations of variation that are self-adjoint, is discussed. As an application of the above theory the following cases are treated: (1) integrals whose extremals are straight lines, (2) integrals whose extremals are semicircles, (3) integrals whose extremals are catenaries.

9. Miss Helen Deutsch: *Note on certain questions of probability.*

This note considers the most probable straight line for a given table of data, and discusses criteria for the line to be unaltered by reversing the selection of the independent variable, also an upper limit for the changes in the constants of the line for given errors in the table. Some curious results of random sampling are also reported.

10. Professor L. L. Dines: *On positive solutions of a system of linear equations.*

This paper presents a simple algorithm for determining whether a given system of linear algebraic equations with real numerical coefficients admits a solution in which each of the unknowns has a *positive* value, and for determining such a solution when one exists. It will appear in the Annals of Mathematics.

11. Professor L. L. Dines: *On completely signed sets of functions.*

A set of functions $f_1(x), f_2(x), \cdots, f_m(x)$, continuous on an interval $a \leqq x \leqq b$, is said to be a *completely signed* set, if there exist 2^m points $x_1, x_2, \cdots, x_{2^m}$, on the interval such that the matrix $\|f_i(x_j)\|$ is a completely signed matrix (as already defined, Annals of Mathematics, vol. 28, p. 42). The author shows that the condition of being completely signed is sufficient (by no means necessary) to ensure the following two properties for the set: (1) every linear combination of the functions changes sign on the interval; (2) there exists a positive continuous function on the interval which is orthogonal to each function of the set. The further remark is made that the notion of a completely signed set may be extended to a set of functions of a general variable. The paper will appear in the Annals of Mathematics.

12. Mr. W. B. Doggett: *On certain caustic surfaces and optical phenomena.*

If the farther half of a spherical flask be silvered, and the flask be nearly filled with water, the observer will see at bottom an apparent bubble or globule of immiscible fluid. Experiment indicates that this is the image of the upper surface of the water, distorted by reflection and refraction; and this conclusion is verified mathematically. Preliminary to this study, there is a discussion of caustic surfaces formed by reflection from spherical and parabolic surfaces.

13. Mr. W. H. Gage: *Asymptotic satellites near the equilibrium point in the isosceles triangle solution of the problem of three bodies (elliptic case).*

If two finite bodies of equal mass move in circles or ellipses about their common center of gravity, then an infinitesimal third body may be initially projected so that it will move in a periodic orbit and remain equidistant from the finite bodies. The paper determines the orbits which are asymptotic to this periodic orbit when the motion of the finite bodies is elliptic.

14. Professor F. L. Griffin: *Points of minimum travel for a distributed population.*

The General Report of the U. S. Census, after defining the center of population, erroneously states that it is the point at which all the people could be assembled with the minimum aggregate travel, going directly. This suggests the problem of determining points of minimum travel for areas of various shapes and for a population whose density varies in an arbitrary manner. Several cases are discussed.

15. Miss Bonnie Haack: *Certain families of curves having a parabola as their envelope.*

This paper formulates the inverse of the envelope problem for families of coplanar curves, and studies certain families of straight lines, circles, and parabolas having a given parabola as their envelope.

16. Mr. A. R. Jerbert: *Projective differential geometry of a system of linear, homogeneous, first-order differential equations.*

This paper studies a system of linear, homogeneous, first-order differential equations in three dependent and one independent variables. A fundamental system of independent solutions can be interpreted as the homogeneous coordinates of three points in a plane. This system of equations defines three curves "to within a projectivity." As the independent variable varies, the points describe three associated curves, corresponding points being those given by the same value of the independent variable. The complete system of invariants of the system is obtained and interpreted as expressing projective properties of the associated curves.

17. Professor R. E. Moritz: *The general solution of a certain Diophantine equation in three unknowns.*

A certain problem in partitions leads to the equations $x-1=kn$, $y=hn$, where k, n, and h are positive integers subject to the condition $(k+1)(n-1)-n^{n-1}(hn+n-1)=0$ but otherwise unrestricted. This paper deals with the general positive integral solutions.

18. Professor R. E. Moritz: *Some new formulas for estimating depreciation of physical properties.*

As is well known, the reducing balance method of computing depreciation results in an excessive reduction of book values during the early years of an organization by creating what amounts to a secret reserve. This paper shows how this objection may be removed by considering the ratio of depreciation as a function of the time and by determining this function with regard to the nature of the property under consideration.

19. Professor R. E. Moritz: *A modification of Glaisher's proof of Sterling's theorem.*

Glaisher's so called elementary proof of Sterling's formula for the approximation of large numbers is open to the objection that it assumes without proof the convergence of a certain double series, and assumes as constant a variable which approaches a constant as a limit. The paper offers an elementary proof of the theorem which is free from these objections.

20. Professor L. I. Neikirk: *A totally discontinuous function.* Preliminary report.

If $f_1(x)$ and $f_2(x)$ are any two functions of x, then the function under discussion will be equal to f_1 at every point of an interval except at an enumerable set of points which are everywhere dense, where it is equal to f_2. Between every two points at which it equals f_1 there is a point at which it equals f_2, and vice versa. An explicit formula is given for this function.

21. Professor F. S. Nowlan: *Arithmetics of rational division algebras of order nine.*

In this paper the integral elements (in the Dickson sense) of a certain rational division algebra D are determined. The basal units of D are given by $y^ix^j(i, j=0, 1, 2)$, where $y^3=\eta\epsilon$ with η the product of integral powers of 7 and rational primes of the form $7m\pm1$ and ϵ is the product of rational primes $7m\pm2$ and $7m\pm3$, with at least one such factor entering to an integral power not divisible by 3. Further, x satisfies the irreducible cyclic equation $w^3+w^2-2w-1=0$. For such value of ϵ of the form $7m\pm1$ there are three sets of integral elements. If, however, ϵ is of the form $7m\pm2$ or $7m\pm3$, then for each of its values there is unique set of integral elements, viz., the set I of all elements with a rational integral coordinates, referred, however, to a transformed set of basal units in case $\eta\neq1$.

22. Professor F. S. Nowlan: *A note on the direct product of a division algebra and a total matric algebra.*

This paper establishes certain theorems concerning an algebra expressible as the direct product of a division algebra and a total matric algebra. In particular, it develops a method for finding the primitive idempotent elements of a total matric algebra.

23. Professor F. S. Nowlan: *A problem related to matric algebras.*

Given the basis of a total matric algebra all similar bases are determined.

24. Miss Dorothory Pennock: *A generalization of the circular and hyperbolic functions of sectorial areas.*

The familiar line definitions are generalized by measuring the sectorial areas from an origin other than the center of the circle or hyperbola. The functions so generalized are studied in detail as to their graphs, periodicity, basic relations, derivatives, integrals, and development in series. A short discussion of similar functions for the parabola is appended.

25. Miss Marguerite Rollins: *Functionally conjugate curves. The case of the harmonic mean.*

A basic reference curve B and a given curve G are selected, and a conjugate curve R is defined as having its radius vector in any direction the harmonic mean of the corresponding radii vectors of B and G. The character of R is studied when B and G are straight lines or circles or certain other conics. When B and G coincide as a single conic, R becomes the polar of the origin.

26. Mr. Hung Shu So: *On the projective determination and classification of conics through given points.*

This paper presents a detailed discussion of the character of conics as determined by given sets of points, with special attention to degenerate forms, and with simple criteria for the selection of points which will determine any desired type of conic, or vice versa.

27. Dr. W. J. Trjitzinsky: *Integrals of non-analytic functions.*

Functions non-analytic in the sense of being non-monogenic do not possess a unique derivative at a point. If C be a closed curve, then under certain conditions $\left|\int_C F(z)dz\right| < 4\sqrt{2}M \cdot A$, where M is the upper bound of the maximum directional derivative on and within C, and A is the area within C. This relation is an analoge of Cauchy's theorem and is proved by using essentially Goursat's method in proving Cauchy's theorem. As

a corollary, we have upper bounds for certain line integrals much closer than the upper bounds given by the ordinary method.

28. Dr. W. J. Trjitzinsky: *Zeros of a function and zeros of its derivatives.*

This paper will appear in full in an early issue of this Bulletin.

29. Dr. W. J. Trjitzinsky: *Relations satisfied by coefficients of periodic solutions.*

Assuming that there is a periodic solution of a linear homogeneous equation with periodic coefficients containing parameters and differentiating the solution with respect to one of the parameters, the derivative is found to be a periodic solution of a certain linear non-homogeneous differential equation. Solving this by the method of variation of parameters and introducing the condition of periodicity, a relation is found satisfied by the coefficients of a periodic solution containing all of them. This method is, in general, complicated, but the difficulty is avoided by using Fite's recent results concerning periodic solutions. The results of the paper were applied to the Mathieu differential equation of the elliptic cylinder and to Hill's equation of the lunar theory.

30. Dr. W. J. Trjitzinsky: *Representation of functions determined by their initial values.* First paper.

Borel, Denjoy, and de la Vallée Poussin have investigated classes of functions more general than the class of analytic functions; they are characterized by the fact that they are determined by the initial values (of the function and its successive derivatives); de la Vallée Poussin gives a practical method for constructing such functions when the initial values are assigned by means of a series $\alpha_1 \cos m_1 x + \alpha_2 \cos m_2 x + \cdots$ where m_i are integers such that $m_{k+1}/m_k > r > 1$ for all k. In this paper it is shown that the functions under consideration are representable by a series at the form $\alpha_1 f(m_1 x) + \alpha_2 f(m_2 x) + \cdots$ where $|f(x)|$, $|f^{(2p)}(x)| < M$ for all p's and all real x's. As a corollary, it is shown that when m_i are restricted in a certain way, an analytic function $F(x)$ is expansible on $(-1, +1)$ in a series $F(x) = \sum_{i=1}^{\infty} \alpha_i f(m_i x)$, where α_i are given by complex integrals. The proof is close to de la Vallée Poussin's proof.

31. Dr. W. J. Trjitzinsky: *Representation of functions determined by their initial values.* Second paper.

It is found that when a function $F(x)$ which is determined by its initial values is represented by the series $\sum_{i=1}^{\infty} \alpha_i f(m_i x)$, the approximation furnished by taking the first n terms of the series is of the order of $1/r^{2n} (r > 1)$. Moreover, it is found that the functions $f(x)$ which are admissible in the above representation may be computed in the form $f(x) = \sum_{j=0}^{\infty} C_j x^{2j}$ where

$C_i=(-1)^i\sum_{k=1}^{\infty}(p_k^{2i}c_k)$ and c_i, p_i are any numbers such that $\sum_{i=0}^{\infty}|c_i|$ converges and $|p_j|\leq 1$; also, it is found that these functions $f(x)$ satisfy on $(-1, +1)$ relations of the form $f(x)=\sum_{i=1}^{\infty}\alpha_i f(m_i x)$, where α_i are independent of the choice of $f(x)$.

32. Dr. W. J. Trjitzinsky: *Polynomials approximating to functions analytic or quasi-analytic.*

Taking de la Vallée Poussin's representation of functions by means of a series $\sum_{i=1}^{\infty}\alpha_i \cos m_i x$, let us take the first n terms of the series i.e., a trigonometrical polynomial of order m_n which approximates to the function. By Fejér's result it would follow that the maximum of the above approximating polynomial is not greater than m_n times its minimum (within a period), and conversely. In this paper it is shown that under certain slight restrictions the above immediate consequence of Fejér's result can be made more precise (though less immediate); namely, a polynomial of order m_n approximating to any of the functions of the class under consideration has its maximum not greater than n times its minimum ($n<m_n$), and conversely.

33. Dr. G. T. Whyburn: *Some theorems on continuous curves.*

In this paper it is shown that if M denotes any plane continuous curve, the following theorems are true. (1) If G denotes the set of all the cut points $[P]$ of M such that P is an irregular point of some sub continuum of M, then each point of G belongs to some simple closed curve in M, and G is countable. (2) The cut point P of M belongs to no simple closed curve in M if and only if it is an end point of every continuum obtained by adding P to some maximal connected subset of $M-P$. (3) In order that M should fail to separate the plane it is necessary and sufficient that every maximal cyclic curve of M should be a simple closed curve plus its interior. (4) A maximal cyclic curve of M is a simple closed curve if and only if it is the common boundary of two complementary domains of M. (5) If M itself is not the boundary of any domain, than in order that every proper subcontinuum of M should be the boundary of a domain it is necessary and sufficient that M should be the sum of three arcs joining two points A and B of M and having no other common points.

34. Dr. G. T. Whyburn: *Concerning closed and connected subsets of the cut points of continua.*

Let M be a bounded continuum and K any closed and connected subset of the cut points of M. The author has shown (Transactions of this Society, vol. 29, 1927, p. 383) that K is an acyclic continuous curve. In this paper it is shown that: (1) K has only a countable number of end points; (2) if AB is any arc of K then $M-AB$ plus some countable subset T of AB is the sum of the elements of a countable upper semi-continuous collection G' of mutually exclusive continua each of which contains exactly

one point of T; furthermore, if G denotes the collection obtained by adding to G' all those *points* of $AB-T$, then M itself is a *simple continuous arc of elements of* G from the element containing A to the one containing B; (3) $M-K$ plus some countable subset H of K is the sum of the elements of a countable upper semi-continuous collection E' of mutually exclusive continua each of which contains exactly one point of H, and if E denotes the collection obtained by adding to E' all those *points* of $K-H$, then M itself is an *acyclic continuous curve of elements of* E whose end elements are the elements of E' which contain end points of K; (4) M is regular at every point of $K-H$.

35. Professor R. M. Winger: *The equianharmonic cubic and its group.*

The equianharmonic cubic admits, as is well known, a monomial group of order 54, which is a subgroup of the Hesse group. A complete system of invariants of this group is obtained and numerous invariant curves are considered. In particular, the pencil of invariant sextics, which contains several interesting members, is discussed and some new properties of the sextic invariant of the Hesse group are found.

36. Mr. John Biggerstaff: *Arithmetic of nonians.*

The writer finds the set of integral elements of the algebra of nonians according to Dickson's set of properties. Application of the theorem that the product of nonions equals the product of the norms of the factors, gives solutions of diophantine problems involving relations of the third degree.

37. Professor J. W. Campbell: *On the strength of guy wires.* Preliminary report.

A solution is obtained for the restoring couple afforded to a tower by a given pair of symmetric guy wires when the tower leans through a small angle in the plane of the wires.

38. Professor R. L. Moore: *A separation theorem.*

In this paper it will be shown that if T is a totally disconnected closed subset of the boundary of a bounded simply connected domain D and there exists a continuum K containing T and such that $K-T$ is a subset of D then there exists a simple closed curve J containing a subset of T and enclosing a part of $K-T$ and such that $(J+T)-T$ is a subset of $D-(K-T)$.

It is also shown that this proposition does not remain true on the omission of "a part of" before "$K-T$" even though T consists of a simple point on the *outer* boundary of D. This corrects a statement made on page 473 (see Theorem 3) of the author's paper concerning the separation of point sets by curves (Proceedings of the National Academy, vol. 11, 1925).

39. Dr. G. T. Whyburn: *Concerning connected and regular point sets.*

This paper will appear in full in an early issue of this Bulletin.

B. A. BERNSTEIN,
Secretary of the Section

BEQUESTS TO THE SOCIETY

At its meeting held on May 6, 1927, the Board of Trustees of the Society adopted the following resolution:

Resolved, that the Treasurer be, and hereby is, instructed to credit to endowment all bequests made to the American Mathematical Society except those specifically designated for other purposes.

Several members have already made provision for bequests to the funds of the Society. The specific action of the Trustees was taken in view of the suggestion of one of these persons that other members wishing to take similar action might know what disposition would be made of unspecified bequests. For such bequests, the following form, which is sufficient in any state, is suggested:

"I give and bequeath to the American Mathematical Society, a corporation incorporated under the laws of the District of Columbia, the sum of ..... dollars."

Still others have written into their wills bequests for specific purposes, such as prizes. Members who desire to make, such bequests are invited to consult the officers of the Society regarding the most urgent needs of mathematics in this country.

W. B. FITE,
Treasurer.

ON THE GEOMETRY OF LINEAR DISPLACEMENT*

BY D. J. STRUIK

1. *Introduction.* The first to consider the displacement of a vector to a point at an infinitesimal distance in a non-euclidean geometry seems to have been L. E. J. Brouwer for the case of space of constant curvature.† Its importance for differential geometry was pointed out for the first time, however, by Levi-Civita (1917. 1),‡ for the case of a hypersurface immersed in an euclidean space, that is, the case of an n-dimensional Riemannian displacement. It was with this paper of Levi-Civita that the new development of differential geometry was started. Weyl (1918. 1 ; 1918. 3) first pointed out the way in which Levi-Civita's results could be generalized and he originated what we call *the Weyl displacement.* Even before Weyl and Levi-Civita, generalizations of analogous character had been suggested by Hessenberg (1916. 1). The geometry of linear displacement was developed in the papers of König (1919. 1 ; 1920. 1), Eddington (1921. 1 ; 1923. 4), Weyl (1918. 1 ; 1918. 3), and Schouten (1922. 1). Schouten gave a general classification of the different geometries defined by linear displacements, which he published later in his book (1924. 9). We adopt in general Schouten's notation.

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* Address delivered at the meeting of this Society in New York, February 26, 1927. The address is elaborated and provided with a bibliography. Papers published in 1927 are not discussed.

† L. E. J. Brouwer, *Het krachtveld der nieteuklidische negatief gekromde ruimten*, Verslagen Akademie Amsterdam, vol. 15 (1906), pp. 75–94 = *The force field of the non-euclidean spaces with negative curvature*, Proceedings Academy Amsterdam, vol. 9 (1906), pp. 116–133. See however also a remark of H. Poincaré in A. Dall' Acqua, *Sulla teoria delle congruenze di curve in una varietà qualunque a tre dimensioni*, Annali di Matematica, (3), vol. 6 (1901), pp. 1–40.

‡ Such references refer to the bibliography at the end of this paper (see p. 558). Independently of Levi-Civita, Schouten obtained analogous results (1918.4).

That euclidean geometry is a special case of Riemannian geometry had been shown by Riemann. It could now be pointed out by König (1919. 1 ; 1920. 1) and Schouten (1923. 7 ; 1924. 9) that affine geometry, the differential geometry of which had been recently developed by Blaschke, Pick, Radon, and others, had the same relation to the affine displacement, introduced by Weyl, as euclidean geometry had to the displacement of Levi-Civita. The question now arose of approaching projective and conformal geometry by the way of linear displacement. Weyl (1921. 2) was successful in this. Cartan, who had given an introduction to the geometry of linear displacement starting from a different point of view in a series of papers (printed in 1922 and 1923), also approached projective and conformal geometries in his own way (papers of 1924). Schouten (1924. 7; 1924. 8) pointed out the relation between Cartan's and König's results.

Another way of attacking the problems of a linear displacement was found by Eisenhart and Veblen (1922. 2). These authors do not start with the displacement, but define it by a set of partial differential equations of the second order representing the paths (geodesic lines) of the displacement. It was especially the affine displacement that was the object of their investigation and of those of T. Y. Thomas and J. M. Thomas (1924–26). They developed further projective and conformal geometries, and also the non-symmetric displacement. One of their results is a theory of differential invariants of these displacements. Another author who studied this part of the theory is Weitzenböck (1920. 2 ; 1922. 13). T. Y. Thomas also gave a new way to introduce projective geometry into the geometry of paths (1925. 13 ; 1926. 11).

Applications of geometries of linear displacements to the theory of relativity were made by Weyl (1918, 1919), Eddington (1921. 1 ; 1923. 4), Einstein (1923. 1 ; 1923. 2 ; 1923. 3), and others (see bibliography).

A recent application of the theory of linear displacements to the theory of semi-simple and simple continuous groups was given by Cartan and Schouten (1925. 3).

2. *The n-Dimensional Manifold and Ricci Algebra.* The geometry of linear displacement starts with the assumption of sets of points which constitute a *domain* of an *n-dimensional manifold.* Such a domain will be denoted by X_n. This assumption implies a set of postulates which are discussed by Veblen (1925. 16, pp. 132–136).* Veblen introduces points as undefined terms and asserts that there exists a way of labelling the points in a biunique way by means of ordered sets of n numbers x^ν, $\nu = 1, 2, \cdots, n$. This is equivalent to the *introduction of an n-cell* as a set of points capable of supporting a euclidean geometry. We then get a set of postulates for an arbitrary geometry of linear displacements, by first giving such a set for a euclidean geometry and then postulating that the geometry under consideration be *analytically related* to that of Euclid. That means, that if A be a point of the euclidean space, and R_A be an n-dimensional region containing A, in which the points are uniquely denoted by coordinates y^ν, and R_P an analogous region containing a point P of X_n, in which the relations of geometry (as the $\Gamma^\nu_{\lambda\mu}$) are analytic functions of the coordinates x^ν, there is an analytic relation

$$y^\nu = f(x^\mu), \qquad (\mu, \nu = 1, 2, \cdots, n), \tag{1}$$

with a unique inverse.

When the coordinates x^ν of X_n undergo an analytic transformation with a unique inverse

$$'x^\nu = \phi^\nu(x^\mu), \qquad (\mu, \nu = 1, 2, \cdots, n) \tag{2}$$

the differentials of these coordinates in general suffer a homogeneous linear transformation

* See also H. Lebesgue, *Sur les correspondances entre les points de deux espaces,* Fundamenta Mathematicae, vol. 2 (1921), pp. 256–285, where other literature (e.g., Cantor, Brouwer) is cited.

$$(3) \qquad {}'dx^\nu = \frac{\partial' x^\nu}{\partial x^\mu} dx^\mu, \qquad \Delta = \left| \frac{\partial' x^\lambda}{\partial x^\mu} \right| \neq 0,$$

and these transformations build up the linear homogeneous or affine group. Hence the infinitesimal region of X_n at a point P can be considered as an infinitesimal affine space E_n in which the differentials can be regarded as representing a set of rectilinear coordinate axes. Thus we can introduce at one point of an X_n the whole algebra of Ricci calculus, which is identical with the theory of invariants and covariants of the linear homogeneous group. Hence we can speak *at one point* of X_n of contravariant, covariant, and mixed quantities (vectors and tensors), the most simple contravariant vector being dx^ν. The three principal invariant operations between these quantities are

(a) transvection (Überschiebung), for example:

$$v_\lambda w^\lambda, \; v_{\lambda\mu\nu} w^\mu, \; R_{\omega\mu\lambda}^{\cdots\quad\omega}, \text{etc.};$$

(b) alternation, for example:

$$v_{[\mu} w_{\lambda]} = \tfrac{1}{2}(v_\mu w_\lambda - v_\lambda w_\mu),$$

$$v_{[\lambda\mu\nu]} = \tfrac{1}{6}(v_{\lambda\mu\nu} - v_{\lambda\nu\mu} + v_{\mu\nu\lambda} - v_{\mu\lambda\nu} + v_{\nu\lambda\mu} - v_{\nu\mu\lambda});$$

(c) mixing, for example:

$$v_{(\mu} w_{\lambda)} = \tfrac{1}{2}(v_\mu w_\lambda + v_\lambda w_\mu),$$

$$v_{(\lambda\mu\nu)} = \tfrac{1}{6}(v_{\lambda\mu\nu} + v_{\lambda\nu\mu} + v_{\mu\nu\lambda} + v_{\mu\lambda\nu} + v_{\nu\lambda\mu} + v_{\nu\mu\lambda}).$$

3. *Linear Displacement.* To begin with, there is nothing to compare the quantities at any point $P(x^\nu)$ with the quantities at another point $Q(x_1^\nu)$. At one point P we can at least compare the lengths of vectors in the same direction (for instance we can define nv^ν as having n times the length of v^ν*), of simple bivectors $v_{[\mu} w_{\lambda]}$ in the same plane direction, etc. But even this fails at different points, and we can only compare

* We shall denote vectors and tensors by their components, and speak e.g. of the *vector* v^ν instead of the *vector* with components v^ν.

the values at P and Q of a given scalar field. In Riemannian geometry we overcome the difficulty by the introduction of a quadratic differential form

$$ds^2 = g_{\lambda\mu}dx^\lambda dx^\mu, \tag{4}$$

which enables us

(a) to define at one point the length of a contravariant vector and the angle between two of them,

(b) to compare these vectors, and hence their lengths and angles at two different points P and P' an infinitesimal distance apart.

This is done by *geodesic parallelism*, introduced by Levi-Civita (1917. 1), by means of which it is possible (see also (Schouten 1918. 4)) to define the *covariant differentials* of a vector field $v^\nu(x^1, \cdots, x^n)$, uniquely defined at P and P':

$$\begin{cases} \delta v^\nu = dv^\nu + \left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\} v^\lambda dx^\mu, \\ \delta w_\lambda = dw_\lambda - \left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\} w_\nu dx^\mu, \end{cases} \tag{5}$$

where the $\left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\}$ depend on $g_{\lambda\mu}$ and $\partial g_{\lambda\mu}/\partial x^\nu$ as follows:

$$\left\{ \begin{matrix} \lambda\mu \\ \sigma \end{matrix} \right\} = g^{\nu\sigma} \left[\begin{matrix} \lambda\mu \\ \nu \end{matrix} \right] = \frac{1}{2} g^{\nu\sigma} \left(\frac{\partial g_{\lambda\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\nu}}{\partial x^\lambda} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu} \right); \tag{6}$$

$g^{\nu\sigma}$ = (minor of $g_{\nu\sigma}$ in determ. of $g_{\lambda\mu}$) ÷ this determ.

When $\delta v^\nu = 0$ or $\delta w_\lambda = 0$, we call the respective vectors at P and P' *parallel*. The formulas (5) define the *Riemannian displacement*.

In an X_n we have no fixed tensor $g_{\lambda\mu}$, but we can still give a definition of parallelism which enables us to retain part of (b). Instead of the $\left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\}$ we therefore introduce, for one set of x^ν, n^3 arbitrary functions $\Gamma^\nu_{\lambda\mu}$ and n^3 arbitrary functions $\Gamma'^\nu_{\lambda\mu}$ and define

$$\begin{cases} \delta v^\nu = dv^\nu + \Gamma^\nu_{\lambda\mu} v^\lambda dx^\mu, \\ \delta w_\lambda = dw_\lambda - \Gamma^\nu_{\lambda\mu} w_\nu dx^\mu, \end{cases} \tag{7}$$

as *covariant differentials* of the vector fields v^ρ and w_λ. It is therefore necessary that a transformation of coordinates (3) imply a transformation of the $\Gamma^\rho_{\lambda\mu}$ and $\Gamma'^\rho_{\lambda\mu}$ as follows:

$$(8)\quad \begin{cases} \Gamma'^{\kappa}_{\omega\pi} = \dfrac{\partial x^\lambda}{\partial' x^\omega}\dfrac{\partial x^\mu}{\partial' x^\pi}\dfrac{\partial' x^\kappa}{\partial x^\rho}\Gamma^\rho_{\lambda\mu} + \dfrac{\partial x^\mu}{\partial' x^\pi}\dfrac{\partial' x^\kappa}{\partial x^\rho}\dfrac{\partial}{\partial x^\mu}\dfrac{\partial x^\rho}{\partial' x^\omega}, \\ {}'\Gamma'^{\kappa}_{\omega\pi} = \dfrac{\partial x^\lambda}{\partial' x^\omega}\dfrac{\partial x^\mu}{\partial' x^\pi}\dfrac{\partial' x^\kappa}{\partial x^\rho}\Gamma'^{\rho}_{\lambda\mu} + \dfrac{\partial x^\mu}{\partial' x^\pi}\dfrac{\partial' x^\kappa}{\partial x^\rho}\dfrac{\partial}{\partial x^\mu}\dfrac{\partial x^\rho}{\partial' x^\omega}, \end{cases}$$

or

$$(9)\quad \begin{cases} \Gamma^\rho_{\lambda\mu}\dfrac{\partial' x^\omega}{\partial x^\rho} = {}'\Gamma^\omega_{\rho\sigma}\dfrac{\partial' x^\rho}{\partial x^\lambda}\dfrac{\partial' x^\sigma}{\partial x^\mu} + \dfrac{\partial^2 x^\omega}{\partial x^\lambda \partial x^\mu}, \\ \Gamma'^{\rho}_{\lambda\mu}\dfrac{\partial' x^\omega}{\partial x^\rho} = {}'\Gamma'^{\omega}_{\rho\sigma}\dfrac{\partial' x^\rho}{\partial x^\lambda}\dfrac{\partial' x^\sigma}{\partial x^\mu} + \dfrac{\partial^2 x^\omega}{\partial x^\lambda \partial x^\mu}. \end{cases}$$

When the Γ are transformed in this way, δv^ρ and δw_λ are transformed as vectors. The transformations of the covariant and of the contravariant vectors are independent of each other. As the formulas (7) for δv^ρ and δw_λ are linear in the dx^ρ, the displacement they define is called the *general linear displacement*. It passes into the Riemannian displacement if a tensor $g_{\lambda\mu}$ exists so that

$$(10)\qquad \Gamma^\rho_{\lambda\mu} = \Gamma'^{\rho}_{\lambda\mu} = \left\{ \begin{matrix} \lambda\mu \\ \rho \end{matrix} \right\},$$

and if this tensor $g_{\lambda\mu}$ is taken as the fundamental tensor (that is, as defining the ds^2 as in (4)). This general linear displacement (7) was first introduced by Schouten (1922. 1; see also 1924. 9). Schouten also gave a set of axioms on which this displacement is founded (1924. 9, pp. 63–64). These ideas had been anticipated by König (1919. 1; 1920. 1), whose conclusions will be considered in §11.

When $\delta v^\rho = 0$, or $\delta w_\lambda = 0$, we say that the corresponding vector is moved *parallel* to itself in the sense of the defined displacement.

From the covariant differentials we pass to the covariant derivatives by the formulas

$$\delta v^\nu = dx^\mu \nabla_\mu v^\nu, \qquad \delta w_\lambda = dx^\mu \nabla_\mu w_\lambda ; \tag{11}$$

hence

$$\begin{cases} \nabla_\mu v^\nu = \dfrac{\partial v^\nu}{\partial x^\mu} + \Gamma^{\nu}_{\lambda\mu} v^\lambda, \\ \nabla_\mu w_\lambda = \dfrac{\partial w_\lambda}{\partial x^\mu} - \Gamma'^{\nu}_{\lambda\mu} w_\nu. \end{cases} \tag{12}$$

The covariant differentials of quantities of higher degree are obtained by assuming the formula, valid for all quantities ϕ and ψ :

$$\delta(\phi\psi) = (\delta\phi)\psi + \phi\delta\psi, \tag{13}$$

so that, for example,

$$\begin{cases} \delta g_{\lambda\mu} = dg_{\lambda\mu} - \Gamma'^{\alpha}_{\lambda\beta} g_{\alpha\mu} dx^\beta - \Gamma'^{\alpha}_{\mu\beta} g_{\lambda\alpha} dx^\beta, \\ \nabla_\mu v^\alpha_{\cdot\beta\gamma} = \dfrac{\partial}{\partial x^\mu} v^\alpha_{\cdot\beta\gamma} + \Gamma^{\alpha}_{\kappa\mu} v^{\kappa}_{\cdot\beta\gamma} - \Gamma'^{\kappa}_{\beta\mu} v^{\alpha}_{\cdot\kappa\gamma} - \Gamma'^{\kappa}_{\gamma\mu} v^{\alpha}_{\cdot\beta\kappa}. \end{cases} \tag{14}$$

The covariant differential of a scalar field $p(x^\nu)$ is the ordinary differential

$$\delta p = dp, \qquad \nabla_\mu p = \frac{\partial p}{\partial x^\mu}. \tag{15}$$

4. *The Fields $C^{\cdots\nu}_{\lambda\mu}$ and $S^{\cdots\nu}_{\lambda\mu}$.* The Γ and Γ' are not tensors, as is shown by (9). However

$$C^{\cdots\nu}_{\lambda\mu} = \Gamma^{\nu}_{\lambda\mu} - \Gamma'^{\nu}_{\lambda\mu} \tag{16}$$

has tensor character (Schouten 1922. 1), as is shown by (9), or by the formula

$$\Delta_\mu A^\nu_\lambda = \frac{\partial}{\partial x^\mu} A^\nu_\lambda + \Gamma^{\nu}_{\alpha\mu} A^\alpha_\lambda - \Gamma'^{\alpha}_{\lambda\mu} A^\nu_\alpha = \Gamma^{\nu}_{\lambda\mu} - \Gamma'^{\nu}_{\lambda\mu}, \tag{17}$$

where

$$A^\nu_\lambda \begin{cases} = 0, & \nu = \lambda \\ = 1, & \nu = \lambda \quad \text{(not summed)} \end{cases} \tag{18}$$

is the unit tensor (often called the Kronecker symbol and denoted by δ_λ^ν *).

It plays a role in the covariant differentiation of a transvection, for example,

$$\delta(v_\lambda w^\lambda) = (\delta v_\lambda) w^\lambda + v_\lambda \delta w^\lambda - C_{\mu\lambda}^{\cdots\nu} v_\nu w^\lambda dx^\mu. \tag{19}$$

When we consider geometries in which the equation

$$v^\lambda w_\lambda = 0 \tag{20}$$

remains invariant under displacement, we need the following form for $C_{\lambda\mu}^{\cdots\nu}$:

$$C_{\lambda\mu}^{\cdots\nu} = C_\mu A_\lambda^\nu. \tag{21}$$

In this case the *incidence* of the point represented by v^ν, and the affine E_{n-1} represented by w_λ, remains invariant under displacement. If not only incidences are invariant, but also the transvection itself, we have $C_\mu = 0$, or

$$C_{\lambda\mu}^{\cdots\nu} = 0. \tag{22}$$

This case (22) is assumed by most authors. The independence of the displacements in the contravariant and covariant cases, characterized by $C_{\lambda\mu}^{\cdots\nu} \neq 0$, is considered only in the papers of König, Schouten, and Dienes (1924. 24). In the following pages we shall asume in general that (22) is satisfied. For this case Dienes considers the integrals of (12), which lead to functions of lines.

In Schouten's† book (1924. 9) formulas for the case $C_{\lambda\mu}^{\cdots\nu} \neq 0$ are given.

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* For vectors and tensors we always use Latin letters in this paper.

† Schouten also deals with another generalization. He admits a definition of covariant vectors by means of an equation

$$'w_\lambda = \tau^{-1} \frac{\partial x^\nu}{\partial' x^\lambda} w_\nu; \quad w_\lambda = \tau \frac{\partial' x^\nu}{\partial x^\lambda} w_\nu,$$

where $\tau(x^\nu)$ is an arbitrary function. For $\tau = 1$ we get the ordinary definition of a covariant vector. The meaning of this definition is trivial in an E_n, but not in an X_n. He calls it the *alteration of covariant measure.* An analogous alteration of contravariant measure is not possible unless the differentials dx^ν cease to be exact differentials.

We now observe that the expressions

$$S_{\lambda\mu}^{\cdots\nu}=\tfrac{1}{2}(\Gamma_{\lambda\mu}^{\nu}-\Gamma_{\mu\lambda}^{\nu}) \tag{23}$$

have tensor character, as is shown by (9) or by the formula

$$\delta_2 d_1x^\nu-\delta_1 d_2x^\nu=2d_1x^\lambda d_2x^\mu S_{\lambda\mu}^{\cdots\nu}. \tag{24}$$

Thus if d_1x^ν is moved parallel to itself (in the sense of the displacement) in the direction d_2x^ν and in the same way d_2x^ν in the direction d_1x^ν we do not get in general a closed quadrilateral. Only when $S_{\lambda\mu}^{\cdots\nu}=0$ (except for quantities of higher order) do we always obtain a closed figure. In the case of $S_{\lambda\mu}^{\cdots\nu}\neq 0$, Eddington speaks of "infinitely crinkled," and Cartan of "torsion." The case $S_{\lambda\mu}^{\cdots\nu}\neq 0$ we will call *non-symmetric*, and the case $S_{\lambda\mu}^{\cdots\nu}=0$ *symmetric*. In this last case

$$\Gamma_{\lambda\mu}^{\nu}=\Gamma_{\mu\lambda}^{\nu}, \tag{25}$$

which expresses the *Christoffel symmetry*. An intermediate case is that in which

$$S_{\lambda\mu}^{\cdots\nu}=S_{[\lambda}A_{\mu]}^{\nu}=(S_\lambda A_\mu^{\nu}-S_\mu A_\lambda^{\nu})/2. \tag{26}$$

Then the quadrilateral of d_1x^ν and d_2x^ν is closed when and only when the directions of d_1x^ν and d_2x^ν lie in the E_{n-1} of S_λ. This case is called *semi-symmetric*. Hessenberg (1916. 1), the first to introduce a geometry belonging to general linear displacement, obtained the non-symmetric case. He then specialized on the symmetric case (1916. 1, pp. 210–211). Afterwards Weyl also came to the symmetric case (1918. 1) in the paper in which he introduced the idea of displacement in the sense in which we use it here. Weyl spoke of an *affine* displacement.

The curl of a vector field w_λ is, in a non-symmetric displacement,

$$\nabla_{[\mu}v_{\lambda]}=\frac{1}{2}\left(\frac{\partial v_\lambda}{\partial x^\mu}-\frac{\partial v_\mu}{\partial x^\lambda}\right)-S_{\lambda\mu}^{\cdots\nu}v_\nu, \tag{27}$$

and hence only depends on the alternating part of the $\Gamma_{\lambda\cdot\cdot}^{\nu}$.

To every asymmetric displacement belongs, in a univocal way, a symmetric one. For let $\Lambda^{\nu}_{\lambda\mu}$ be the symmetric part of $\Gamma^{\nu}_{\lambda\mu}$:

$$\Lambda^{\nu}_{\lambda\mu} = \tfrac{1}{2}(\Gamma^{\nu}_{\lambda\mu} + \Gamma^{\nu}_{\mu\lambda}), \tag{28}$$

or

$$\Gamma^{\nu}_{\lambda\mu} = \Lambda^{\nu}_{\lambda\mu} + S^{\cdot\cdot\nu}_{\lambda\mu}, \tag{29}$$

then $\Lambda^{\nu}_{\lambda\mu}$ is transformed in the same way (8, 9) as $\Gamma^{\nu}_{\lambda\mu}$, and thus defines a displacement. Also by writing

$$\begin{aligned}\delta v^{\nu} &= dv^{\nu} + \Gamma^{\nu}_{\lambda\mu} v^{\lambda} dx^{\mu} = (dv^{\nu} + \Gamma^{\nu}_{\lambda\mu} v^{\lambda} dx^{\mu}) \\ &\quad + S^{\cdot\cdot\nu}_{\lambda\mu} v^{\lambda} dx^{\mu},\end{aligned} \tag{30}$$

and considering the vector character of $S^{\cdot\cdot\nu}_{\lambda\mu} v^{\lambda} dx^{\mu}$, we see the vector character of

$$^{*}\delta v^{\nu} = dv^{\nu} + \Lambda^{\nu}_{\lambda\mu} v^{\lambda} dx^{\mu}. \tag{31}$$

This remark was made by J. M. Thomas (1926.2).

5. *Introduction of a Metric.* Again taking $C^{\cdot\cdot\nu}_{\lambda\mu} = 0$, we may observe that, in a general linear displacement, the differential of a tensor $g^{\lambda\mu}$ gives an affinor $Q^{\cdot\lambda\mu}_{\mu}$ of third degree:

$$\begin{cases} \delta g^{\lambda\nu} = Q^{\cdot\lambda\nu}_{\mu} dx^{\mu}, \\ \nabla_{\mu} g^{\lambda\nu} = Q^{\cdot\lambda\nu}_{\mu}. \end{cases} \tag{32}$$

Then, differentiating the covariant tensor $g_{\lambda\mu}$ belonging to $g^{\lambda\mu}$, if we assume that its rank is n, we get by (6),

$$\begin{cases} \delta g_{\lambda\mu\nu} = -\, g_{\lambda\alpha} g_{\nu\beta} Q^{\cdot\alpha\beta}_{\mu} dx^{\mu}, \\ \nabla_{\mu} g_{\lambda\nu} = -\, g_{\lambda\alpha} g_{\nu\beta} Q^{\cdot\alpha\beta}_{\mu}. \end{cases} \tag{33}$$

Suppose now, that the $\Gamma^{\nu}_{\lambda\mu}$ can be so chosen that there exists at least one tensor (of rank n) $g_{\lambda\mu}$ so that

$$Q^{\cdot\lambda\nu}_{\mu} = Q_{\mu} g^{\lambda\nu}, \tag{34}$$

hence

$$\begin{cases} \delta g^{\lambda\nu} = Q_{\mu} g^{\lambda\nu} dx^{\mu}, \\ \nabla_{\mu} g^{\lambda\nu} = Q_{\mu} g^{\lambda\nu}. \end{cases} \tag{35}$$

In that case also

$$\delta g_{\lambda\nu} = -\, Q_\mu g_{\lambda\nu} dx^\mu \tag{36}$$

and we speak of a *conformal* displacement (Schouten 1924.9, p. 72). That this geometry is possible is seen from the general theorem which we will also state for the case of $C_{\lambda\mu}^{\cdots\nu} \neq 0$, *that it is possible to define in a unique way every set of* $\Gamma_{\lambda\mu}^{\nu}$ *and* $\Gamma_{\lambda\mu}'^{\nu}$, *and hence every linear displacement, by two arbitrary affinor fields* $C_{\lambda\mu}^{\cdots\nu}$ *and* $S_{\lambda\mu}^{\cdots\nu}$ *and a field* $Q_\mu^{\cdot\lambda\nu}$ *defined as the derivative of a tensor field* $g^{\lambda\nu}$ *of rank n.*

The proof can be given by the explicit elimination of $\Gamma_{\lambda\mu}^{\nu}$ and $\Gamma_{\lambda\mu}'^{\nu}$ from the equations expressing $C_{\lambda\mu}^{\cdots\nu}$, $S_{\lambda\mu}^{\cdots\nu}$ and $Q_\mu^{\cdot\lambda\nu}$ (and $g^{\lambda\nu}$) in terms of the $\Gamma_{\lambda\mu}^{\nu}$ and $\Gamma_{\lambda\mu}'^{\nu}$. The explicit formula for the case $C_{\lambda\mu}^{\cdots\nu} = 0$ is

$$\Gamma_{\lambda\mu}^{\nu} = \left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\} + T_{\lambda\mu}^{\cdots\nu}, \tag{37}$$

where the Christoffel symbol must be taken as belonging to $g^{\lambda\nu}$ and

$$\left\{ \begin{aligned} T_{\lambda\mu}^{\cdots\nu} = {} & \tfrac{1}{2}(g_{\lambda\alpha} Q_\mu^{\cdot\alpha\nu} + g_{\mu\alpha} Q_\lambda^{\cdot\nu\alpha} - g^{\nu\gamma} g_{\mu\alpha} g_{\lambda\beta} Q_\gamma^{\cdot\alpha\beta}) \\ & + S_{\lambda\mu}^{\cdots\nu} - g^{\nu\beta}(g_{\lambda\alpha} S_{\beta\mu}^{\cdots\alpha} + g_{\mu\alpha} S_{\beta\lambda}^{\cdots\alpha}). \end{aligned} \right. \tag{38}$$

This theorem was first given by Schouten (1922. 1; see also 1924. 9, p. 72–74).

If now we take $Q_\mu^{\cdot\lambda\nu} = Q_\mu g^{\lambda\nu}$ in (37) and (38), we can construct a set of $\Gamma_{\lambda\mu}^{\nu}$ which define a conformal displacement.

In a conformal geometry every tensor

$$'g_{\lambda\mu} = \sigma g_{\lambda\mu}, \tag{39}$$

where σ is an arbitrary function of x^ν, has a differential of the form (35) or (36). For we have

$$\left\{ \begin{aligned} \delta' g_{\lambda\nu} &= -\, dx^\mu \left(\frac{\partial \sigma}{\partial x^\mu} - \sigma Q_\mu \right) g_{\lambda\nu} \\ &= -\left(Q_\mu - \frac{\partial \log \sigma}{\partial x^\mu} \right) g_{\lambda\nu} dx^\mu \\ &= -\, Q_\mu' g_{\lambda\nu} dx^\mu, \end{aligned} \right. \tag{40}$$

where

$$Q'_\mu = Q_\mu - \frac{\partial \log \sigma}{\partial x^\mu}. \tag{41}$$

We take one of these tensors $g_{\lambda\mu}$ and define by means of it the angle between two directions v^ν, w^ν :

$$\cos\theta = \frac{g_{\lambda\mu}v^\lambda w^\mu}{\sqrt{g_{\lambda\mu}v^\lambda v^\mu}\sqrt{g_{\lambda\mu}w^\lambda w^\mu}}, \tag{42}$$

which is invariant under (39). We then call the tensors (39) the *fundamental tensors* of the displacement. We have proved by (40), that when *one tensor* $g_{\lambda\mu}$ *has a differential of the form* (36), *every tensor* $\sigma g_{\lambda\mu}$ *has a differential of this form.* It can also be shown *that only the fundamental tensors* $\sigma g_{\lambda\mu}$ *possess this property* (Schouten 1924.9, p. 220).

If in particular Q_μ is a gradient vector, there is, by (41), one fundamental tensor $g_{\lambda\mu}$ uniquely determined so that

$$\delta g_{\lambda\mu} = 0. \tag{43}$$

In this case the displacement is called *metrical.* We may remark *that one and only one linear displacement is determined by a field* $S_{\lambda\mu}^{\ \ .\nu}$ *alternating in* λ *and* μ *and a tensor field* $g_{\lambda\mu}$ *of rank n, when* $\Gamma^\nu_{\lambda\mu} = S_{\lambda\mu}^{\ \ .\nu}$ *and* $\nabla_\omega g_{\lambda\mu} = 0$. This is a special case of the preceding theorem.

When this tensor $g_{\lambda\mu}$ is taken as fundamental tensor, we can define length and angle by means of (42) and

$$ds^2 = g_{\lambda\mu}dx^\lambda dx^\mu. \tag{44}$$

Length and angle of vectors remain invariant under a metrical displacement, and only under such a displacement, as is easily shown. When in particular $\Gamma^\nu_{\lambda\mu} = \Gamma^\nu_{\mu\lambda}$, the conformal displacement passes into the *displacement of Weyl,* introduced by him (1918. 1, 3 ; see also 1918. 2), and the metrical displacement into the *displacement of Levi-Civita,* (1917. 1), which is the foundation of Riemannian geometry. Indeed, in this case (37) passes into

$$\Gamma^{\nu}_{\lambda\mu} = \left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\}. \tag{45}$$

A simple example of a metrical displacement which is not symmetric was given by Cartan (1924.5) and Hessenberg (1925. 24). Take a sphere and a set of meridians on it. Define the displacement by the assumption that a contravariant vector is moved parallel to itself when the angle with the meridians is always the same. Length and angle remain invariant, hence the displacement is metrical. But it is not the displacement of Levi-Civita, hence there must be an affinor $S^{\cdots\nu}_{\lambda\mu} \neq 0$. It can easily be seen, by (66), that this metric on the sphere is even holonomous (see §8). The geodesic lines of this displacement are the loxodromics of the sphere. For other examples see Cartan (1926. 15). For the relation to the parallelism of Clifford see Cartan (1924. 5).

Necessary conditions that a symmetric system of $\Gamma^{\nu}_{\lambda\mu}$ may be of the form (45) or of the form of Weyl geometry were given by Eisenhart-Veblen (1922. 2). Extension of their methods can be found in Schouten (1925. 18) and Veblen-Thomas (1923. 14).

6. *Geometry of Paths.* Two other ways have been pointed out by which to approach the linear displacements (with $C^{\cdots\nu}_{\lambda\mu} = 0$), the geometry of paths developed by Veblen and Eisenhart (1922. 2), and the linear displacements of points instead of vectors developed by Cartan (1923 and later, see for example, 1925. 1). The geometry of paths does not start with the displacement itself, but with the geodesic lines. A geodesic line arises when a contravariant vector or a covariant $(n-1)$-vector moves parallel to itself in its own direction. Hence the equations of a *contravariant geodesic line* are

$$v^{\mu}\nabla_{\mu}v^{\nu} = \alpha v^{\nu}, \tag{46}$$

where v^{ν} means a contravariant vector tangent to the curve $x^{\nu} = x^{\nu}(t)$, and α is a function of the x^{ν}. The equations of a *covariant geodesic line* are

$$v^{\mu}\nabla_{\mu}v_{\lambda_1 \cdots \lambda_{n-1}} = \beta v_{\lambda_1 \cdots \lambda_{n-1}}, \tag{47}$$

where $v_{\lambda_1 \ldots \lambda_{n-1}}$ is a covariant $(n-1)$-vector tangent to the curve, and β is a function of the x^ν. A covariant geodesic line is, however, also a contravariant one, whenever $C^{\cdots\nu}_{(\lambda\mu)} = C_{(\lambda} A^{\nu}_{\mu)}$ (Friedmann-Schouten 1924. 11). Since we suppose $C^{\cdots\nu}_{\lambda\mu} = 0$, we have only to deal with the equation (46), or

$$\frac{d^2x^\nu}{dt^2} + \Gamma^{\nu}_{\lambda\mu}\frac{dx^\lambda}{dt}\frac{dx^\mu}{dt} = \alpha\frac{dx^\nu}{dt}. \tag{48}$$

From (48) we see that the equations of geodesics only depend on the symmetric part $\Lambda^{\nu}_{\lambda\mu}$ of $\Gamma^{\nu}_{\lambda\mu}$ (see end §4). The coefficient depends on the choice of the parameter t. We can make $\alpha = 0$, if we introduce a new parameter l defined, by the equation

$$l = C_1 + C_2 e^{\int \alpha\, dt}\, dt, \tag{49}$$

with two constants of integration C_1 and C_2. We thus get the equations of the geodesics:

$$\frac{d^2x^\nu}{dl^2} + \Lambda^{\nu}_{\lambda\mu}\frac{dx^\lambda}{dl}\frac{dx^\mu}{dl} = 0. \tag{50}$$

Hence *we have found on a geodesic line a natural scale of measurement determined but for the choice of an origin and a multiplicative factor* (Eisenhart 1922. 5).

The equation (50) is the starting point for the geometry of paths. The integral curves are the natural generalization of the straight lines of euclidean geometry and of the geodesics of Riemannian geometry. To start with this equation, means to consider the geometry of linear displacements as a geometric theory of a particular kind of differential equations of the more general class

$$\frac{d^2x^\nu}{ds^2} = \phi^\nu\left(x, \frac{dx}{ds}\right), \tag{51}$$

where ϕ^ν is an arbitrary function of the x^ν and its derivatives. (For $n = 2$, see Cartan 1924. 4, p. 232.) From this point of

view the study of equations (50) is a continuation of studies started by Pascal and others.*

7. *Normal Coordinates.* It is now possible to define a special set of coordinates at each point of X_n by means of the paths defined by (50). Take a point $P(x_0^\nu)$ and a direction $(dx^\nu/ds)_0 = v_0^\nu$ through it.

Differentiations of (50) with respect to the natural parameter gives

$$\frac{d^3x^\nu}{ds^3} + \Lambda^\nu_{\lambda_1\lambda_2\lambda_3}\frac{dx^{\lambda_1}}{ds}\frac{dx^{\lambda_2}}{ds}\frac{dx^{\lambda_3}}{ds} = 0, \tag{52}$$

$$\frac{d^4x^\nu}{ds^4} + \Lambda^\nu_{\lambda_1\lambda_2\lambda_3\lambda_4}\frac{dx^{\lambda_1}}{ds}\frac{dx^{\lambda_2}}{ds}\frac{dx^{\lambda_3}}{ds}\frac{dx^{\lambda_4}}{ds} = 0, \tag{53}$$

where

$$\Lambda^\nu_{\lambda_1\cdots\lambda_p\omega} = \frac{\partial}{\partial x}(\omega\Lambda_{\lambda_1\cdots\lambda_p}) - p\Lambda^\nu_{\alpha(\lambda_1\cdots\lambda_{p-1}}\Lambda^\alpha_{\lambda_p\omega)}. \tag{54}$$

Then we can obtain the expansion of the coordinates of a geodesic line;

$$\begin{aligned} x^\nu = x_0^\nu + v_0^\nu t - \frac{1}{2!}(\Lambda^\nu_{\lambda\mu})_0 v_0^\lambda v_0^\mu t^2 \\ - \frac{1}{3!}(\Lambda^\nu_{\lambda\mu\nu})_0^\lambda v_0 v_0^\mu v_0^\nu t^3 - \cdots, \end{aligned} \tag{55}$$

which converges for all values of t if we assume the $\Lambda^\nu_{\lambda\mu}$ to possess derivatives of all orders at x_0^ν, $|v_0^\nu|$ to be finite, and

$$|\Lambda^\nu_{\lambda_1\cdots\lambda_p}| < G, \tag{56}$$

for all values of p, where G is a given constant.

If we write

$$v_0^\nu t = y^\nu, \tag{57}$$

* E. Pascal, *Grundlagen für eine Theorie der Systeme totaler Differentialgleichungen zweiter Ordnung*, Mathematische Annalen, vol. 54 (1901), pp. 400–416; see also Comptes Rendus, vol. 130 (1900), pp. 645–647.

(56) passes into

$$x - x_0^r = y^r - \frac{1}{2!}(\Lambda^r_{\lambda\mu})_0 y^\lambda y^\mu - \frac{1}{3!}(\Lambda^r_{\lambda\mu\nu})_0 y^\lambda y^\mu y^\nu - \cdots . \tag{58}$$

Since the functional determinant $|\partial x^r/\partial y^\lambda| \neq 0$, we can solve these equations with respect to y^r, and we get the convergent expansion

$$\left\{ \begin{aligned} y^r &= (x^r - x_0^r) + \frac{1}{2!}(\Omega^r_{\lambda\mu})_0 (x^\lambda - x_0^\lambda)(x^\mu - x_0^\mu) \\ &+ \frac{1}{3!}(\Omega^r_{\lambda\mu\nu})_0 (x^\lambda - x_0^\lambda)(x^\mu - x^\mu)(x^\nu - x_0^r) + \cdots , \end{aligned} \right. \tag{59}$$

where

$$\left\{ \begin{aligned} \Omega^r_{\lambda\mu} &= \Lambda^r_{\lambda\mu}, \\ \Omega^r_{\lambda\mu\omega} &= \Lambda^r_{\lambda\mu\omega} + 3\Omega^r_{\alpha(\lambda}\Lambda^\alpha_{\mu\omega)}, \\ \Omega^r_{\lambda_1\lambda_2\lambda_3\lambda_4} &= \Lambda^r_{\lambda_1\lambda_2\lambda_3\lambda_4} + 4\{\Omega^r_{\alpha(\lambda_1}\Lambda^\alpha_{\lambda_2\lambda_3\lambda_4)} \\ &\quad + \Omega^r_{\alpha\beta}\Lambda^\beta_{(\lambda_1\lambda_2}\Lambda^\alpha_{\lambda_3\lambda_4)} + \Omega^r_{\alpha(\lambda_1\lambda_2}\Lambda^\alpha_{\lambda_3\lambda_4)}\}. \end{aligned} \right. \tag{60}$$

Through (58) and (59) new coordinates are defined in a finite region around $P(x_0^r)$, the *normal coordinates*. They are the natural generalization of Riemann's normal coordinates for Riemannian geometry,* and were first introduced by Veblen (1922. 3). Under an arbitrary transformation (4) of the x the y^r are transformed linearly and homogeneously, as follows from (57), and precisely as the components of a contravariant vector. In an E_n the (y^r) would represent the radius vector of a point with respect to P, and then the *normal coordinates give a representation of the X_n on the E_n at P.*

The ordinary derivatives of a quantity at P with respect to the normal coordinates at P are components of tensors.

* See also G. D. Birkhoff, *Relativity and Modern Physics*, Cambridge, Harvard University Press, 1923. Birkhoff calls these coordinates *Riemannian* coordinates (p. 118). He uses the term *Normal coordinates* (p. 124) for another purpose.

For example, when $v_{\lambda\mu\nu}$ is an affinor, and $\bar{v}_{\lambda\mu\nu}$ are its components in the system of normal coordinates, then the equations

$$(61)\qquad \begin{cases} w_{\lambda\mu\nu\omega} = \left(\dfrac{\partial \bar{v}_{\lambda\mu\nu}}{\partial y^{\omega}}\right)_0, \\ w_{\lambda\mu\nu\omega_1\omega_2} = \left(\dfrac{\partial^2 \bar{v}_{\lambda\mu\nu}}{\partial y^{\omega_1}\partial y^{\omega_2}}\right)_0 \end{cases}$$

define a set of functions of x^{ν} which satisfy the equations

$$(62)\qquad \begin{cases} w_{\lambda\mu\nu\omega} = \nabla_{\omega} v_{\lambda\mu\nu}, \\ w_{\lambda\mu\nu\omega_1\omega_2} = \nabla_{\omega_1}\nabla_{\omega_2} v_{\lambda\mu\nu}. \end{cases}$$

The first derivatives with respect to the normal coordinates are identical with the symmetric part of the covariant derivative. A more complete set of formulas (62) is given by Veblen-Thomas (1923. 14, pp. 573–576).

As is easily found from (50) and (58) or from (59), $\Lambda^{\nu}_{\lambda\mu}$ vanishes for normal coordinates at P. This takes place for every transformation which has two terms in common with the right side of (59), in particular for

$$(63)\qquad x^{\nu} - x^{\nu}_0 = y^{\nu} - \frac{1}{2!}(\Lambda^{\nu}_{\lambda\mu})_0 y^{\lambda} y^{\mu}.$$

Such a system of coordinates is called *geodesic* (Weyl 1918. 2, p. 101 of 4th ed.) or *path coordinates* (Eisenhart, 1923. 11). The expansion for $\Lambda^{\nu}_{\lambda\mu}$ in normal coordinates is

$$(64)\qquad \Lambda^{\nu}_{\lambda\mu} = M^{\nu}_{\lambda\mu\alpha} y^{\alpha} + \frac{1}{2!} M^{\nu}_{\lambda\mu\alpha\beta} y^{\alpha} y^{\beta} + \cdots.$$

where

$$(65)\qquad M^{\nu}_{\lambda\mu\alpha_1\alpha_2\cdots\alpha_p} = \left(\frac{\partial^p \Lambda^{\nu}_{\lambda\mu}}{\partial y^{\alpha_1}\partial y^{\alpha_2}\cdots\partial y^{\alpha_p}}\right)_0.$$

The $\Lambda^{\nu}_{\lambda\mu}$ have no tensor character.

8. *Curvature.* If P and Q are points of a curve and v^{ν} is a field of vectors, the integral $\int_P^Q \delta v^{\nu}$ taken along s is the difference

of v^ν at Q and the vector obtained by parallel moving of the vector at P along s to Q. This integral depends not only on P and Q but also on the curve s. When it does not depend on the curve, but only on P and Q for every pair of points P and Q, the X_n is called *holonomous* (Cartan 1925. 2, see 1923. 21). If s is closed and $Q=P$, we find in general a difference after the return of the vector at P to P. This difference can be calculated easily if the closed curve s is infinitely small. We find for the difference Dv^ν at P

$$Dv^\nu = R_{\omega\mu\lambda}^{\cdots\nu} v^\lambda f^{\mu\omega} d\sigma, \tag{66}$$

where $f^{\omega\mu}$ is a unit bivector in the plane element which contains s (but for differential elements of higher than the first order), $d\sigma$ is the area of the element (both in such a measure that $f^{\omega\mu}\, d\sigma$ represents the bivector of the surface element determined by s), and

$$R_{\omega\mu\lambda}^{\cdots\nu} = \frac{\partial}{\partial x^\mu}\Gamma_{\omega\lambda}^{\nu} - \frac{\partial}{\partial x^\omega}\Gamma_{\mu\lambda}^{\nu} + \Gamma_{\kappa\mu}^{\nu}\Gamma_{\lambda\omega}^{\kappa} - \Gamma_{\kappa\omega}^{\mu}\Gamma_{\lambda\mu}^{\kappa}. \tag{67}$$

In the same way, we find

$$Dw_\lambda = - R_{\omega\mu\lambda}^{\cdots\nu} w_\nu f^{\omega\mu} d\sigma. \tag{68}$$

The symbol $R_{\omega\mu\lambda}^{\cdots\nu}$ is an affinor, the so called *curvature affinor* of the X_n. It always appears in investigations where the alternating part of the second derivative is considered, with $C_{\lambda\mu}^{\cdots\nu}=0$, according to the formulas

$$\begin{cases} 2\nabla_{[\omega}\nabla_{\mu]} v^\nu = - R_{\omega\mu\lambda}^{\cdots\nu} v^\lambda + 2S_{\omega\mu}^{\cdots\alpha}\Delta_\alpha v^\nu, \\ 2\nabla_{[\omega}\nabla_{\mu]} w_\lambda = R_{\omega\mu\lambda}^{\cdots\nu} w_\nu + 2S_{\omega\mu}^{\cdots\alpha}\nabla_\alpha w_\lambda. \end{cases} \tag{69}$$

The interpretation (66), which for the case of a Riemannian geometry was given by Levi-Civita (1917. 1), was given for Weyl's geometry by Weyl (1918. 2). Compare for discussion and exact demonstration Synge (1923.12) and T. Y. Thomas (1926. 3). See also Eddington (1923. 4, p. 68, and p. 214 of second edition).

The effect of the operator $\nabla_{[\omega}\nabla_{\mu]}$ on quantities of higher degree is seen (see (14)) from the example

$$2\nabla_{[\omega}\nabla_{\mu]}v_{\alpha}^{\cdot\beta\gamma} = R_{\omega\mu\alpha}^{\cdots\rho}\, v_{\rho}^{\cdot\beta\gamma} - R_{\omega\mu\rho}^{\cdots\beta}\, v_{\alpha}^{\cdot\rho\gamma} - R_{\omega\mu\rho}^{\cdots\gamma}\, v_{\alpha}^{\cdot\beta\rho} + S_{\omega\mu}^{\cdot\cdot\delta}\, \nabla_{\delta} v_{\alpha}^{\cdot\beta\gamma}. \tag{70}$$

The affinor $R_{\omega\mu\lambda}^{\cdots\nu}$ is subject to the following identities. From the definition equation (66) follows the *first identity*:

$$R_{\omega\mu\lambda}^{\cdots\nu} = -R_{\mu\omega\lambda}^{\cdots\nu}. \tag{71}$$

Again, by calculating $\nabla_{[\omega}\nabla_{\mu}\nabla_{\lambda]}$, first by applying ∇_{ω} to $\nabla_{\mu}v_{\lambda}$ and then alternating, secondly by applying $\nabla_{[\omega}\nabla_{\mu]}$ to v_{λ} and then alternating, we find the *second identity*:

$$R_{[\omega\mu\lambda]}^{\cdots\nu} = 4S_{[\omega\mu}^{\cdots\alpha} S_{\lambda]\alpha}^{\cdots\nu} + 2\nabla_{[\omega} S_{\mu\lambda]}^{\cdots\nu}. \tag{72}$$

For a symmetric displacement this passes into the form

$$R_{[\omega\mu\lambda]}^{\cdots\nu} = 0, \tag{73}$$

or

$$R_{\omega\mu\lambda}^{\cdots\nu} + R_{\mu\lambda\omega}^{\cdots\nu} + R_{\lambda\omega\mu}^{\cdots\nu} = 0. \tag{74}$$

Moreover, by calculating $\nabla_{[\omega}\nabla_{\mu]}\, g_{\lambda\nu}$, first by applying ∇_{ω} to $\nabla_{\mu}\, g_{\lambda\nu}$, given as in (33), and then alternating, secondly by applying $\nabla_{[\omega}\nabla_{\mu]}$ to $g_{\lambda\nu}$ as in (70), we get the *third identity*:

$$R_{\omega\mu\lambda}^{\cdots\alpha}\, g_{\alpha\nu} + R_{\omega\mu\nu}^{\cdots\alpha}\, g_{\alpha\lambda} = 2\nabla_{[\omega} Q_{\mu]\lambda\nu}, \tag{75}$$

where

$$Q_{\mu\lambda\nu} = -\, g_{\lambda\alpha} g_{\nu\beta} Q_{\mu}^{\cdot\alpha\beta}.$$

For a Riemannian geometry, this passes into

$$R_{\omega\mu\lambda\nu} = -\, R_{\mu\omega\lambda\nu}. \tag{76}$$

In a Riemannian geometry, we can deduce from the three identities the fourth:

$$R_{\omega\mu\lambda\nu} = R_{\lambda\nu\omega\mu}. \tag{77}$$

The four identities of $R_{\omega\mu\lambda}^{\cdots\nu}$ for a Riemannian geometry have been given by Riemann (in his Paris memoir, 1861); for more general cases they appear in Weyl, Eddington, Veblen, Eisenhart and Schouten's papers, in Schouten's book also for the case of $C_{\lambda\mu}^{\cdots\nu} \neq 0$ (1924. 9, p. 87). The dependence of the fourth identity on the other three was first pointed out by Ricci* (see also Hessenberg 1916. 1). It also follows as a corollary from a general result of T. Y. Thomas (1926. 10).

For the derivative of $R_{\omega\mu\lambda}^{\cdots\nu}$ also an identity exists, which can be obtained, for example, by calculating $\nabla_{[\xi}\nabla_{\omega}\nabla_{\mu]}v_{\lambda}$ first by applying ∇_{ξ} on $\nabla_{[\omega}\nabla_{\mu]}v_{\lambda}$, secondly by applying $\nabla_{[\xi}\nabla_{\omega]}$ on $\nabla_{\mu}g_{\lambda}$ and then alternating on ξ, ω, μ. For $C_{\lambda\mu}^{\cdots\nu} = 0$ it is

$$\nabla_{[\xi}R_{\omega\mu]\lambda}^{\cdots\nu} = -2S_{[\omega\mu}^{\cdots\alpha}R_{\xi]\omega\lambda}^{\cdots\nu}. \tag{78}$$

This identity is called *Bianchi's identity*. For a symmetric displacement it becomes

$$\nabla_{[\xi}R_{\omega\mu]\lambda}^{\cdots\nu} = 0. \tag{79}$$

Bianchi published this identity in 1902 for a Riemannian geometry,† though it can be found in earlier papers of Padova (1889), who refers to a letter of Ricci.‡ Bach (1921. 3) proved it for a geometry of Weyl, Veblen (1922. 3) and Schouten (1923. 6) for symmetric displacements; Weitzenböck (1923. 25, p. 357) deduced (78). The general formulas for $C_{\lambda\mu}^{\cdots\nu} \neq 0$ can be found in Schouten (1924. 9, p. 91). (See also Schouten-Struik 1924. 10.) Cartan obtained (78) in another way (1923. 21, vol. 40, p. 373).

From $R_{\omega\lambda\mu}^{\cdots\nu}$, the following affinors can be deduced:

$$\begin{cases} R_{\mu\lambda} = R_{\omega\mu\lambda}^{\cdots\omega} \\ V_{\omega\mu} = R_{\omega\mu\lambda}^{\cdots\lambda} \\ F_{\mu\lambda} = R_{[\mu\lambda]} = R_{\omega[\mu\lambda]}^{\cdots\omega}. \end{cases} \tag{80}$$

* G. Ricci, *Sulla determinazione di varietà, dotate di proprietà intrinseche date a priori*, Rendiconti dei Lincei, (5), vol. 19^{II} (1910), pp. 85–90.

† L. Bianchi, *Sui simboli a quattro indice e sulla curvatura di Riemann*, Rendiconti dei Lincei, (5), vol. 11^{II} (1902), pp. 3–7.

‡ E. Padova, *Sulle deformazioni infinitesime*, Rendiconti dei Lincei, (4), vol. 5^{I} (1889), pp. 174–178.

In a Riemannian geometry $V_{\omega\mu}$ and $F_{\mu\lambda}$ vanish, and $R_{\mu\lambda}$ is symmetric. We then can build up a scalar

$$R = g^{\lambda\mu}R_{\mu\lambda} = g^{\lambda\mu}g^{\omega\nu}R_{\omega\mu\lambda\nu}. \tag{81}$$

From the differential equations for the quantities (80), obtained by contraction from the identity of Bianchi, we mention only

$$\nabla_\mu G^{\mu\lambda} = 0, \qquad G_{\mu\lambda} = R_{\mu\lambda} - \tfrac{1}{2}Rg_{\mu\lambda}, \tag{82}$$

which only holds for a Riemannian geometry. It was obtained by Ricci* and is one of the principal formulas of Einstein's theory of relativity. The equation

$$R_{\omega\mu\lambda}^{\cdots\lambda} = V_{\omega\mu} = 0 \tag{83}$$

has a simple geometric meaning. It means that the displacement conserves the volume of an n-dimensional element after a circuit along a closed curve. Hence we can introduce the conception of *volume* in the X_n, and there exists an integral invariant

$$\int U_{\lambda_1\cdots\lambda_n} f^{\lambda_1\cdots\lambda_n} d\tau_n, \tag{84}$$

where $U_{\lambda_1\cdots\lambda_n}$ is an arbitrary n-vector, and $f^{\lambda_1\cdots\lambda_n}$ a unit n-vector, $d\tau_n = dx^1 \cdots dx^n$ (Eddington 1921. 1, page 111; Eisenhart 1922.5; and Veblen 1923. 8). In normal coordinates, hence for a symmetric displacement, the curvature affinor arises from the expansion of $\Lambda_{\lambda\mu}^{\nu}$ in normal coordinates (64) in the form

$$(R_{\omega\mu\lambda}^{\cdots\nu})_0 = \left(\frac{\partial}{\partial y^\mu}\Lambda_{\lambda\omega}^{\nu}\right)_0 - \left(\frac{\partial}{\partial y^\omega}\Lambda_{\lambda\mu}^{\nu}\right)_0. \tag{85}$$

From this property the two first identities and the identity of Bianchi are easily proved (Veblen-Thomas 1923. 14, pp.

* G. Ricci, *Sulle superficie geodetiche in una varietà qualunque e in particolare nella varietà a tre dimensioni*, Rendiconti dei Lincei, (5), vol. 12 (1903), pp. 409–420.

579–580). The holonomous case is characterized by $R^{\cdots\nu}_{\omega\mu\lambda}=0$. An example is the euclidean case.

9. *Projective Transformations of the Paths.* We consider a symmetric displacement with $C^{\cdots\nu}_{\lambda\mu}=0$ (affine displacement). We then can ask if there are transformations of the $\Gamma^{\nu}_{\lambda\mu}$, which leave the paths invariant. They are found to be

$$'\Gamma^{\nu}_{\lambda\mu} = \Gamma^{\nu}_{\lambda\mu} + A^{\nu}_{\lambda}p_{\mu} + A^{\nu}_{\mu}p_{\lambda}, \tag{86}$$

where p_λ is an arbitrary covariant vector. (Weyl 1921. 2; Veblen-Thomas 1926. 4, p. 281.) Such transformations are called *projective*. Under such a transformation the curvature affinor $R^{\cdots\nu}_{\omega\mu\lambda}$ passes into

$$\begin{aligned} {}'R^{\cdots\nu}_{\omega\mu\lambda} = R^{\cdots\nu}_{\omega\mu\lambda} &- 2A^{\nu}_{\lambda}\nabla_{[\omega}p_{\mu]} \\ &+ 2A_{[\omega}\nabla_{\mu]}p_{\lambda} + 2A^{\nu}_{[\omega}p_{\mu]}p_{\lambda}. \end{aligned} \tag{87}$$

From (87) follows

$${}'R^{\cdots\lambda}_{\omega\mu\lambda} = {}'V_{\omega\mu} = V_{\omega\mu} - 2(n+1)\nabla_{[\omega}p_{\mu]}. \tag{88}$$

This equation is completely integrable on account of Bianchi's identity; hence we find

$$\nabla_{[\xi}R^{\cdots}_{\omega\mu]\lambda}v^{\lambda} = \Delta_{[\xi}v_{\omega\mu]} = 0, \tag{89}$$

so that we see, that *an affine displacement can always be projectively transformed into a displacement which preserves volume* (Eisenhart, 1922. 5; Veblen, 1922. 6; 1922. 13).

The projective theory of affine displacements is the theory of those tensors which are unchanged by the transformation (86). See, however, a remark of Veblen-Thomas (1926. 4). One of these tensors, the *projective curvature tensor*, was found by Weyl (1921. 2); it is

$$P^{\cdots\nu}_{\omega\mu\lambda} = R^{\cdots\nu}_{\omega\mu\lambda} - 2P_{[\omega\mu]}A^{\nu}_{\lambda} + 2A^{\nu}_{[\omega}P_{\mu]\lambda}, \tag{90}$$

where

$$P_{\mu\lambda} = -\frac{1}{n^2-1}(nR_{\mu\lambda} + R_{\lambda\mu}). \tag{91}$$

For $n=1$ and $n=2$, $P^{\cdots\nu}_{\omega\mu\lambda}$ vanishes identically. It satisfies, for $n>2$, the identities

$$(92)\qquad \begin{cases} P^{\cdots\nu}_{\omega\mu\lambda} = -P^{\cdots\nu}_{\mu\omega\lambda}, \\ P^{\cdots\nu}_{[\omega\mu\lambda]} = 0, \\ \nabla_{[\xi}P^{\cdots\nu}_{\omega\mu]\lambda} = \dfrac{1}{n-2}\nabla_\alpha A^\nu{}_{[\omega}P^{\cdots\omega}_{\mu\xi]\lambda}, \end{cases}$$

which were given by Schouten (1924.9, pp. 131–132).

To obtain the projective tensors in a systematic way, we start with the observation of T. Y. Thomas (1925. 7) that the functions

$$(93)\qquad \Pi^\nu_{\lambda\mu} = \Gamma^\nu_{\lambda\mu} - \frac{1}{n+1}(A^\nu_\lambda\Gamma^\alpha_{\mu\alpha} + {}^*A^\nu_\mu\Gamma^\alpha_{\lambda\alpha})$$

are unaltered by transformations (86). The $\Pi^\nu_{\lambda\mu}$ build up a *projective displacement*, and are subject to the identity

$$(94)\qquad \Pi^\alpha_{\lambda\alpha} = 0.$$

The paths of the projective displacements can now be made solutions of the equations

$$(95)\qquad \frac{d^2x^\nu}{dp^2} + \Pi^\nu_{\lambda\mu}\frac{dx^\lambda}{dp}\frac{dx^\mu}{dp} = 0,$$

where p is the projective parameter, defined but for a transformation

$$(96)\qquad {}'p = A\int\Delta^{2/(n+1)}dp + B,$$

where $\Delta = |\partial' x^\lambda/\partial x^\mu|$ (see (3)), and where A and B are constants. This transformation involves Δ, and the same holds for that of $\Pi^\nu_{\lambda\mu}$:

$$(97)\qquad \Pi^\nu_{\lambda\mu}\frac{\partial' x^\omega}{\partial x^\nu} = {}'\Pi^\omega_{\rho\sigma}\frac{\partial' x^\rho}{\partial x^\lambda}\frac{\partial' x^\sigma}{\partial x^\mu} + \frac{\partial^2 x^\omega}{\partial x^\lambda\partial x^\mu} + \frac{1}{n+1}\left(\frac{\partial' x^\omega}{\partial x^\lambda}\frac{\partial \log\Delta^{-1}}{\partial x^\mu} + \frac{\partial' x^\omega}{\partial x^\mu}\frac{\partial\log\Delta^{-1}}{\partial x^\lambda}\right).$$

The problem of finding covariants of linear displacements may be stated as the study of the integrability conditions of the equations (9) for $\Gamma^{\nu}_{\lambda\mu}$. In the same way the problem of projective equivalence of two affine displacements may be reduced to the study of the integrability conditions of the equations (97). This has been done by Veblen-Thomas (1925. 8; 1926. 4), anticipated by J. M. Thomas (1925. 9) and T. Y. Thomas (1925. 7). They introduce *projective normal coordinates*, and use them for the deduction of projective covariants. These, however, need not have tensor character. This is only the case if $P^{\cdots\nu}_{\omega\mu\lambda}=0$. As the main result of this study we may consider a set of *completeness theorems* and the theorem on *projective equivalence* of two affine displacements.

For transformations with $\Delta=1$ the $\Pi^{\nu}_{\lambda\mu}$ transform like the $\Gamma^{\nu}_{\lambda\mu}$ and the p is uniquely determined. This case is called *equi-projective* (Thomas 1925. 7).

If the affine displacement can be projectively transformed so that $R^{\cdots\nu}_{\omega\mu\lambda}=0$, we call this case *projective-euclidean*. We then obtain from (87)

$$R^{\cdots\nu}_{\xi\omega\mu\lambda} = 2P_{[\omega\mu]}A^{\nu}_{\lambda} - 2A^{\nu}_{[\omega}P_{\mu]\lambda}. \tag{98}$$

The integrability conditions, that (91) may exist under the auxiliary condition (98), reduce to

$$\nabla_{[\omega}P_{\mu]\lambda} = 0. \tag{99}$$

When however, we apply the identity of Bianchi to (98), we obtain

$$\nabla_{[\xi}P_{\lambda\mu]} = \tfrac{1}{3}(n-2)\nabla_{[\xi}P_{\mu]\lambda}, \tag{100}$$

which for $n>2$ is identical with (99). In this case the integrability conditions of (91) are a consequence of (98). Then, as $P_{\mu\lambda}$ is defined by (91), (98) passes (see (90)) into

$$P^{\cdots\nu}_{\omega\mu\lambda} = 0. \tag{101}$$

For $n=2$ condition (98) is always satisfied. Hence we have the theorem that *an affine displacement for $n>2$ is projective-*

euclidean when and only when $P^{\cdots\nu}_{\omega\mu\lambda}=0$, *and for* $n=2$ *when and only when* $P_{\omega\mu}$, *defined by* (99) *satisfies* (91) (Weyl 1921. 2).

10. *Conformal Transformation of a Riemannian Manifold.* A close analogy exists between the projective transformations of affine manifolds and the conformal transformations

$$'g_{\lambda\mu} = \sigma g_{\lambda\mu}, \tag{102}$$

where σ is an arbitrary function of the x^ν, in a Riemannian manifold (102) implies

$$'g^{\lambda\mu} = \sigma^{-1} g^{\lambda\mu}. \tag{103}$$

Then (see (86)) we have

$$'\left\{\begin{matrix}\lambda\mu\\ \nu\end{matrix}\right\} = \left\{\begin{matrix}\lambda\mu\\ \nu\end{matrix}\right\} + \frac{1}{2}(A^{\nu}_{\lambda}s_\mu + A^{\nu}_{\mu}s_\lambda - g^{\nu\alpha}g_{\mu\lambda}s_\alpha), \tag{104}$$

where

$$s_\mu = \frac{\partial \log\sigma}{\partial x^\mu}. \tag{105}$$

These conformal transformations leave the angle of two vectors invariant. Under them the curvature affinor $R^{\cdots\nu}_{\omega\mu\lambda}$ passes into the form

$$'R^{\cdots\nu}_{\omega\mu\lambda} = R^{\cdots\nu}_{\omega\mu\lambda} - g_{\omega\lambda}s_{\mu\alpha}g^{\alpha\nu} + g_{\lambda\mu}s_{\omega\alpha}g^{\alpha\nu}, \tag{106}$$

where (see (87))

$$s_{\mu\alpha} = 2\nabla_\mu s_\alpha - s_\mu s_\alpha + \tfrac{1}{2}\, s_\beta s^\beta g_{\mu\alpha}. \tag{107}$$

Equations (104) were obtained by Fubini*; equation (107) by Weyl (1918. 3).

We can ask if a conformal transformation can preserve the geodesic lines. Then, by (86) and (104) we have

$$\tfrac{1}{2}\, A^{\nu}_{\lambda}s_\mu + \tfrac{1}{2}\, A^{\nu}_{\mu}s_\lambda - \tfrac{1}{2}\, s^\nu g_{\lambda\mu} = A^{\nu}_{\lambda}p_\mu + A^{\nu}_{\mu}p_\lambda, \tag{108}$$

* G. Fubini, *Sulla teoria degli spazi che ammettono un gruppo conforme*, Atti di Torino, vol. 38 (1903), pp. 262–276.

which cannot be fulfilled, as is easily shown by contraction with respect to ν, λ; and by transvection with $g^{\lambda\mu}$; for both operations give

$$s_\mu = \frac{2(n+1)}{n} p_\mu = -\frac{4}{n-2} p_\mu, \tag{109}$$

which has no solutions for integer n. Hence we have the theorem *that a Riemannian displacement is fully determined when we know its geodesic lines and the fundamental tensor but for a factor* (Weyl 1921. 2).

Hence projective and conformal transformations exclude each other. Eyraud (1925. 4) has shown how a general deformation of the paths can be decomposed into two deformations, one a generalization of the projective, the other of the conformal transformation.

The conformal theory of Riemannian displacements is the theory of those tensors which are unchanged by the transformation (102) (or 104). (See, however, Veblen-Thomas' remark 1926. 4.) One of these, the conformal curvature tensor, was found by Weyl (1918. 2); it is

$$C_{\omega\mu\lambda\nu} = R_{\omega\mu\lambda\nu} - \frac{2}{n-2}(g_{\omega\lambda}L_{\mu\nu} - g_{\mu\lambda}L_{\omega\nu}), \tag{110}$$

when

$$L_{\mu\lambda} = -R_{\mu\lambda} + \frac{1}{2(n-1)} R g_{\mu\lambda}. \tag{111}$$

For $n=1$, $n=2$, and $n=3$, $C_{\omega\mu\lambda\nu}$ vanishes identically. It satisfies, for $n>3$, the four identities

$$\left\{\begin{array}{l} C_{\omega\mu\lambda\nu} = -C_{\mu\omega\lambda\nu}, \\ C_{[\omega\mu\lambda]\nu} = 0, \\ C_{\omega\mu\lambda\nu} = -C_{\omega\mu\nu\lambda}, \\ C_{\omega\mu\lambda\nu} = C_{\lambda\nu\omega\mu}. \end{array}\right. \tag{112}$$

To obtain the conformal affinor in a systematic way, we start with the remark of J. M. Thomas (1925. 12) that the functions

$$(113)\quad Z^{\nu}_{\lambda\mu} = \left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\} - \frac{1}{n}\left[A^{\nu}_{\lambda} \left\{ \begin{matrix} \mu\alpha \\ \alpha \end{matrix} \right\} + A^{\nu}_{\mu} \left\{ \begin{matrix} \lambda\alpha \\ \alpha \end{matrix} \right\} \right] + \frac{1}{n} g^{\nu\alpha} g_{\lambda\mu} \left\{ \begin{matrix} \alpha\beta \\ \beta \end{matrix} \right\}$$

are unaltered by transformations (102). The $Z^{\nu}_{\lambda\mu}$ build up a *conformal displacement*, and are subject to the identity

$$(114)\quad Z^{\alpha}_{\lambda\alpha} = 0.$$

The paths of the conformal displacement are given by

$$(115)\quad \frac{d^2x^{\nu}}{dc^2} + Z^{\nu}_{\lambda\mu} \frac{dx^{\lambda}}{dc} \frac{dx^{\mu}}{dc} = 0,$$

where c is the *conformal parameter*. For a discussion of this equation in the same way as (95) see J. M. Thomas (1925. 12). See also T. Y. Thomas (1925. 15; 1926. 14).

If the Riemannian displacement can be conformally transformed so that $R^{\cdot\cdot\cdot\nu}_{\omega\mu\lambda} = 0$, this case is called *conformal-euclidean*. We then obtain, from (104),

$$(116)\quad R_{\omega\mu\lambda\nu} = g_{\omega\lambda} s_{\mu\nu} - g_{\mu\lambda} s_{\omega\nu}.$$

The integrability conditions that (116) may exist under the auxiliary condition (107), reduce to

$$(117)\quad \nabla_{[\omega} s_{\mu]\lambda} = 0.$$

When, however, we apply the identity of Bianchi to (116), we obtain

$$(n-3)\nabla_{[\omega} s_{\mu]\lambda} = 0,$$

which, for $n > 3$, is identical with (117). Then, as $s_{\mu\lambda}$ is defined by (107), (116) passes into (see (110))

$$C_{\omega\mu\lambda\nu} = 0.$$

For $n=3$, condition (116) is always satisfied. We thus have (see the end of §9) *that a Riemannian displacement for $n>3$ is conformal-euclidean when and only when $C_{\omega\mu\lambda\nu}=0$, and for $n=3$, when and only when $s_{\omega\mu}$, defined by* (116), *satisfies* (117) (Schouten 1921. 7; for $n=3$ Cotton,* Finzi†).

11. *The Point of View of Point Displacement.* The ideas expressed above all start with the consideration of *vectors* in the E_n at one point P of the X_n that are compared to *vectors* in an E_n at another point P' at infinitesimal distance. Cartan has made the remark that this is not the only way to proceed. We can also start with the *points* in the E_n at a point P of X_n and define our displacement as a law that enables us to compare the *points* at P with those at P'. In this manner, Cartan first derived in a new way the asymmetric affine displacement (1922. 8), and later on could come to generalizations of projective and conformal geometry. We limit ourselves here to an exposition of Cartan's ideas for an affine displacement, and only give some outlines of his further generalizations. Consider, for this (Cartan 1923. 21, the notation is chiefly based on Schouten 1924. 8 and 1926. 12), the E_n of the dx^r at a point P of the X_n and the E_n' at P'. Then the formulas give a law of comparison of the *free* vectors E_n and E_n'. We get a law of comparison of the *points* at E_n and E_n' if we also know how a definite point of E_n passes into a definite point of E_n'. Let us now first obtain in this new way the affine displacements. Introduce in the E_n *homogeneous* coordinates. Every set u^α, $\alpha=0, 1, \cdots, n+1$, abbreviated $\bar{u}$, determines a *point* by its proportion. Then a fundamental $(n+1)$-cell is defined by $n+1$ points $\bar{u}_0, \bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$. Let $\bar{u}_0$ be P itself, and let $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$ be the points at infinity at n oblique axes issuing from P. If

* E. Cotton, *Sur les variétés à trois dimensions*, Annales de Toulouse, (2), vol. 1 (1899), pp. 385–438; also Thèse, Paris, 1899, 61 pp.

† A. Finzi, *Le ipersuperficie a tre dimensioni che si possono rappresentare conformemente sullo spazio euclideo*, Atti, Istituto Veneto, (8), vol. 5(=62), (1902–03), pp. 1049–1062.

we now assume an affine correspondence between E_n at P and E_n' at P', we get our linear connection defined by

$$\delta \bar{u}_\lambda = \Gamma^{\kappa}_{\lambda\mu} \bar{u}_\kappa dx^\mu, \quad (\lambda, \mu, \kappa = 1, \cdots, n). \tag{118}$$

If we now fix the correspondence of P to its image in E' by the equation

$$\delta \bar{u}_0 = - \bar{u}_\mu dx^\mu, \tag{119}$$

a linear connection is determined:

$$\begin{cases} \nabla_\mu v^\alpha = \dfrac{\partial v^\alpha}{\partial x^\mu} + \Lambda^{\alpha}_{\beta\mu} v^\beta, \\ \nabla_\mu w_\alpha = \dfrac{\partial w_\alpha}{\partial x^\mu} - \Lambda^{\beta}_{\alpha\mu} w_\beta, \qquad (\alpha, \beta = 0, 1, \cdots, n), \end{cases} \tag{120}$$

where

$$\begin{cases} \Lambda^{\nu}_{\lambda\mu} = \Gamma^{\nu}_{\lambda\mu}, \\ -\Lambda_{\lambda 0} = A_\mu \begin{cases} = 1, & \nu = \mu, \\ = 0, & \nu \neq \mu, \end{cases} \\ \Lambda^{0}_{\alpha\mu} = 0. \qquad (\lambda, \mu, \nu = 1, \cdots, n). \end{cases} \tag{121}$$

A special case of (119) was treated by Hlavatý (1924. 21). These covariant differentials are mixed quantities, because the differentials dx^μ and the vectors $\bar{u}$ belong to different manifolds. In X_n, however, a displacement is not fixed. But if we assume

$$2\nabla_{[\mu} w_{\lambda]} = \frac{\partial w_\lambda}{\partial x^\mu} - \frac{\partial w_\mu}{\partial x^\lambda} \tag{122}$$

which means that this displacement is at least symmetric, we can define $\nabla_{[\omega}\nabla_{\mu]} v^\nu$ and $\nabla_{[\omega}\nabla_{\mu]} v_\lambda$ and obtain a curvature quantity

$$\mathfrak{A}^{\cdots\beta}_{\omega\mu\alpha} = \frac{\partial}{\partial x^\mu}\Lambda^{\beta}_{\alpha\omega} - \frac{\partial}{\partial x^\omega}\Lambda^{\beta}_{\alpha\mu} + \Lambda^{\beta}_{\delta\mu}\Lambda^{\delta}_{\alpha\mu} - \Lambda^{\beta}_{\delta\omega}\Lambda^{\delta}_{\alpha\mu}, \tag{123}$$
$$(\omega, \mu = 1, \cdots, n\,;\ \alpha, \beta, \delta = 0, 1, \cdots, n).$$

For this tensor we have the first identity and Bianchi's identity

$$(124)\qquad \begin{cases} \mathfrak{A}_{\omega\mu\alpha}^{\cdots\beta} = -\mathfrak{A}_{\mu\omega\alpha}^{\cdots\beta} \\ \nabla_{[\xi}\mathfrak{A}_{\omega\mu]\alpha}^{\cdots\beta} = 0. \end{cases}$$

$\mathfrak{A}_{\omega\mu\alpha}^{\cdots\beta}$ can be derived from $R_{\omega\mu\lambda}^{\cdots\nu}$ and $S_{\mu\lambda}^{\cdots\nu}$. We have

$$(125)\qquad \begin{cases} \mathfrak{A}_{\omega\mu\lambda}^{\cdots\nu} = R_{\omega\mu\lambda}^{\cdots\nu}\ ; \quad \mathfrak{A}_{\omega\mu\lambda}^{\cdots 0} = 0 \\ \mathfrak{A}_{\omega\mu 0}^{\cdots\nu} = -2S_{\omega\mu}^{\cdots\nu}\ ; \quad \mathfrak{A}_{\omega\mu 0}^{\cdots 0} = 0. \end{cases}$$

By means of the formula $Dv^\beta = f^{\omega\mu}\, d\sigma\, \mathfrak{A}_{\omega\mu\alpha}^{\cdots\beta} v^\alpha$, we see that a symmetric displacement is characterized by the fact that $\bar{u}_0$ returns in its own place after a transport along a closed circuit. This is another statement of the fact expressed by formula (24), that in the non-symmetric case the manifold is "infinitely crinkled."

If $R_{\omega\mu\alpha}^{\cdots\gamma} = 0$, we get an E_n. In this case the displacement passes *into the correspondence of all points of the E_n into themselves.*

This last remark leads toward a generalization for the projective and conformal cases, which can now be treated in a way analogous to the preceding paragraph (Cartan 1923. 20; 1924. 4; 1924. 3). The difference is that the E_n at a point P of an X_n must be considered as a projective or conformal space. By the introduction of homogeneous coordinates, projective space is equivalent to affine space of $n+1$ dimensions; by the introduction of tetracyclical, pentaspherical, etc., coordinates, conformal space is equivalent to affine space of $n+2$ dimensions. We thus come to the following generalization: Associate to every point P of an X_n an E_{n+k}, $k=0, 1, 2, \cdots$ and generalize the method of the preceding paragraph to this case. We then have three kinds of quantities:

(a) Quantities in X_n, defined for the E_n of the dx^ν.

(b) Quantities in the E_{n+k} defined in every E_{n+k} with respect to a system of $n+k$ covariant and $n+k$ contravariant fundamental vectors.

(c) Quantities obtained by summing and multiplying the quantities of first and second kind.

We obtain a displacement if we determine how the quantities of the second kind of the E_{n+k} at P correspond to those of the E_{n+k} at a neighboring point P'. We thus obtain a method of treating the results of point displacements by the methods of vector displacement, and find a kind of geometry previously introduced, at least in general, by König (1919. 1; 1920. 1).

Cartan, using his point displacement, now proceeds in an analogous way as in the affine case. He deduces a curvature tensor for the projective and the conformal displacement

$$\mathfrak{P}_{\omega\mu\alpha}^{\cdots\varepsilon}; \qquad \begin{matrix}\omega,\mu = 1, \cdots, n \text{ in } X_n; \\ a,c = 0,1, \cdots, n \text{ in } E_{n+1},\end{matrix} \tag{126}$$

and

$$\mathfrak{C}_{\omega\mu ac}; \qquad \begin{matrix}\omega,\mu = 1, \cdots, n \text{ in } X_n; \\ a,c = 0,1, \cdots, n+1 \text{ in } E_{n+2}.\end{matrix} \tag{127}$$

The relation between point displacements of Cartan and the vector displacements are given by the theorems:

If in an X_n an affine displacement is fixed but for transformations that leave the paths invariant, there is one and only one symmetric projective point connection with the same geodesic lines, for which

$$\sum_k \mathfrak{P}_{kli}^{\cdots k} = 0; \qquad k,l = 1, \cdots, n.$$

Moreover, we have

$$\mathfrak{P}_{kli}^{\cdots m} = P_{kli}^{\cdots m}.$$

The special coordinate system in X_n with respect to which their components are taken is one for which the given affine displacement preserves volume (Cartan 1924. 4, Schouten 1924. 8).

If in an X_n, $n \geqq 3$, Riemann displacement is fixed but for conformal displacements, there is one and only one symmetric conformal point connection with the orthogonality fixed by $g_{\mu\nu}$, for which $\sum_k \mathfrak{C}_{klik} = 0$; $k, l = 1, \cdots, n$. Moreover, $\mathfrak{C}_{klim} = C_{klim}$.

The coordinate system in X_n with respect to which their components are taken is an orthogonal system (Cartan 1923. 20). Cartan's results are discussed from the point of view of vector displacement by Schouten (1926. 12), where also some general remarks on their relation toward the group program of Felix Klein's Erlanger address can be found. A good introduction to Cartan's conceptions is given by his papers of a more general character (1924. 5; 1925. 1).

12. *Manifolds Immersed in Manifolds with a Linear Displacement.* An important reason for the study of X_m, $m < n$, in an X_n with a linear displacement was the desire to have a treatment of affine differential geometry, developed by Blaschke and others,* on the basis of linear displacement. König (1919. 1) pointed out the way, and showed that this affine geometry was the geometry of a linear displacement with

$$(128) \qquad C_{\lambda\mu}^{\cdots\nu} = 0, \quad \Gamma_{\lambda\mu}^{\nu} = \Gamma_{\mu\lambda}^{\nu}, \quad R_{\alpha\mu\lambda}^{\cdots\nu} = 0.$$

Schouten (1923. 7; 1924. 9) took up König's suggestion and gave an elaborate study of this subject. Cartan (1924. 22; 1924. 1; 1923. 21) studied the same subject independently of König. Schouten starts with an X_n and a displacement for which $C_{\lambda\mu}^{\cdots\nu} = 0$, $\Gamma_{\lambda\mu}^{\nu} = \Gamma_{\mu\lambda}^{\nu}$. Consider now an X_{n-1} immersed in the X_n, and a point P of X_{n-1}. Let v^ν be a vector field in X_n; if a vector v^ν is not situated in the tangential E_{n-1} at P, we cannot speak without further assumption of the *v^ν-component* of v^ν in E_{n-1}. This is only possible after the

* See W. Blaschke, *Vorlesungen über Differentialgeometrie*, II, *Affine Differentialgeometrie*, Berlin, Springer, 1923.

introduction of a special direction at P not situated in E_{n-1}, the *pseudonormal* direction. As a covariant vector can be represented by a set of two parallel E_{n-1}, we see that a covariant vector determines, by its intersection with the tangential E_{n-1}, a covariant vector in this E_{n-1} (representable by two parallel E_{n-2}), without any further assumption. After the introduction of a pseudonormal direction, however, it is possible to associate to a covariant vector field in X_{n-1} also a covariant vector field in X_n.

In the same way, we can consider tensors, especially the covariant derivatives of scalars and vectors. Let B^{α}_{β} be the unit tensor of the X_{n-1}, t_λ the tangential covariant vector of the X_{n-1}, and n^ν a vector in the pseudonormal direction. If we now assume $t_\lambda n^\lambda = 1$, which does not fix the scale of measure of t_λ, we have between B^ν and the unit tensor A^{ν}_{λ} of X_n the relation

$$B^{\nu}_{\lambda} = A^{\nu}_{\lambda} - t_\lambda n^\nu. \tag{129}$$

We now define, ∇_μ being the differential operator with respect to the X_n, a linear displacement in the X_{n-1} by means of the equations

$$\begin{cases} {}'\nabla_\mu v^\nu = B^{\alpha}_{\mu} B^{\nu}_{\beta} \nabla_\alpha v^\beta \\ {}'\nabla_\mu w_\lambda = B^{\alpha}_{\mu} B^{\beta}_{\lambda} \nabla_\alpha w_\beta. \end{cases} \tag{130}$$

Even before introducing a normal vector, we may define the derivative of a scalar field

$$'\nabla_\mu p = B^{\alpha}_{\mu} \nabla_\alpha p. \tag{131}$$

The displacement thus defined is also affine (see (17) and (27), since we have

$$\begin{cases} {}'\nabla_\mu B^{\nu}_{\lambda} = B^{\alpha}_{\mu} B^{\beta}_{\lambda} B^{\delta}_{\epsilon} \nabla_\alpha B^{\nu}_{\beta} = - B^{\nu}_{\mu} B^{\beta}_{\lambda} B^{\nu}_{\delta} \nabla_\alpha t_\beta n^\delta = 0, \\ {}'\nabla_{[\omega} {}'\nabla_{\mu]} p = {}'\nabla_{[\omega} (B^{\alpha}_{\mu]} \nabla_\alpha p) = B^{\alpha}_{\mu} \nabla_{[\omega} {}'\nabla_{\alpha]} p \\ \qquad\qquad = B^{\alpha}_{\mu} B^{\beta}_{\omega} \nabla_{[\beta} \nabla_{\alpha]} p = 0, \end{cases} \tag{132}$$

We thus have the fundamental theorem :

If we define at every point of the X_{n-1} a pseudonormal direction, we can define an affine displacement in the X_{n-1} in

such a way that the covariant derivative of a quantity in X_{n-1} is the X_{n-1}-component of the covariant derivative with respect to the X_n.

The displacement in X_{n-1} is called *induced*. We now can induce the following tensor:

$$h_{\mu\lambda} = B^{\alpha}_{\mu} B^{\beta}_{\lambda} \nabla_{\alpha} t_{\beta}, \tag{133}$$

which, by choice of the part of t_β outside of the X_{n-1}, can be made symmetric. Change of the scale of measure for t_λ of the form $'t_\lambda = \sigma t_\lambda$, changes the tensor $h_{\mu\lambda}$ conformally:

$$'h_{\mu\lambda} = \sigma B^{\alpha}_{\mu} B^{\beta}_{\lambda} \nabla_{\alpha} t_{\beta} + B^{\alpha}_{\mu} B^{\beta}_{\lambda} t_{\beta} \nabla_{\alpha} \sigma = \sigma h_{\mu\lambda}. \tag{134}$$

This tensor is perfectly determined after choice of the scale of measure t_λ. In this case a Riemannian displacement is fixed in the X_{n-1}, if $h_{\lambda\mu}$ has the rank $n-1$, and then we take $h_{\lambda\mu}$ as fundamental tensor.

If $P_{\omega\mu\lambda}^{\cdot\cdot\nu} = 0$ in the X_n we can show that

$$T_{\lambda\mu\nu} = T_{\lambda\mu}^{\cdot\cdot\alpha} h_{\alpha\nu} \tag{135}$$

(see (38)) is symmetric, and also that (see (32))

$$T_{\lambda\mu\nu} = \tfrac{1}{2} Q_{\lambda\mu\nu}, \quad h^{\mu\nu} Q_{\lambda\mu\nu} = 0.$$

This condition of apolarity (135) is one of the signs that for the case of ordinary affine differential geometry we have in $h_{\mu\lambda}$ and $Q_{\lambda\mu\nu}$, the so-called first and second fundamental tensors. This geometry is obtained, when the X_n becomes an E_n, through the condition $R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = 0$. The theory of X_{n-1} in X_n and also X_k in X_n $(k < n-1)$ can be developed on this basis. We can obtain, for example, the generalizations of the formulas of Gauss and Codazzi for these cases.

The case of $\Gamma^{\nu}_{\lambda\mu} \neq \Gamma^{\nu}_{\mu\lambda}$ for X_k in X_n was treated by Hlavatý (1926. 13).

Another treatment of these geometries was given by Cartan (1923. 21, vol. 40, p. 403; Cartan has many results not mentioned here). Moreover, Schouten (1924. 9) developed the theory of X_k in X_n for a Weyl geometry.

Frenet formulas for curves in a general linear displacement and a Weyl displacement, but with fixed $g_{\lambda\mu}$, were found by Juvet (1921. 5 ; 1921. 6 ; 1924. 13 ; 1925. 20). Schouten gave the general formulas (1924. 9); see also Juvet (1924. 14).

Eisenhart discussed parallel vector fields in an affine displacement (1922. 4). He calls the vectors of a field *parallel* to one another, if a scalar α exists, so that we have, independent of the direction of displacement dx^r,

$$dx^\lambda \nabla_\lambda \alpha v^r = 0.$$

13. *Linear Displacements and Continuous Groups.* If we consider the ∞^r transformations of an r-parametric continuous group in an X_n as points of an X_r, we obtain the *group-manifold*, in which the parameters are coordinates. Let the transformations be

$$(136)\quad {}'x^r = {}'x^r(x^1, \cdots, x^n, a^1, \cdots, a^r), \quad (\nu = 1, \cdots, n);$$

then the a^λ, $\lambda = 1, \cdots, r$, are the coordinates of X_r. The transformation (136) may be denoted by T_a.

If T_a corresponds with the point a^r and T_{a+da} with the point $a^r + da^r$, then the infinitesimal transformation $T_{a+da}T_a^{-1}$ corresponds to the line element da^r. The line element db^r at point b^r is characterized by $T_{b+db}T_b^{-1}$. This correspondence of both transformations can be expressed by

$$T_{b+db}T_b^{-1} = T_{a+da}T_a^{-1}.$$

This defines a linear displacement, through which a vector at a point corresponds to one and only one vector at another point. Hence $R_{\omega\mu\lambda}{}^r = 0$. It can be shown, however, that $\Gamma^r_{\lambda\mu} \neq \Gamma_{\mu\lambda}$. Another displacement with these two properties can be defined by

$$T_b^{-1} T_{b+db} = T_a^{-1} T_{a+da}.$$

The study of simple and semi-simple groups* under this point of view was started by Cartan-Schouten (1925. 3). Related to these investigations is a paper of Eisenhart (1925. 10).

* First introduced by E. Cartan, *La structure des groupes de transformations continus et la théorie du trièdre mobile*, Bulletin des Sciences Mathématiques, (2), vol. 34 (1910), pp. 250–283.

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THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY

CONCERNING THE BOUNDARIES OF DOMAINS OF A CONTINUOUS CURVE*

BY W. L. AYRES

We shall consider a space M consisting of all the points of a plane continuous curve M†, and all point sets mentioned are assumed to be subsets of M. A connected set of points D of M is said to be an *M-domain* if $M-D$ is closed. The set of all limit points of D which do not belong to D is called the *M-boundary* of D. If D is an M-domain, D' denotes the M-boundary of D, and $\overline{D}$ denotes the set $D+D'$. The M-boundary of an M-domain D is closed but not necessarily connected, even if D is simply connected, as we may easily show by examples. If N is a continuum, a maximal connected subset of $M-N$ is called a *complementary M-domain* of N.

THEOREM I.‡ *Every closed and connected subset of the M-boundary of a complementary M-domain of a continuous curve N is a continuous curve.*

PROOF. Let K denote a closed and connected subset of the M-boundary of a complementary M-domain D of N. Suppose K is not connected im kleinen. Then there exist§ two concentric circles C_1 and C_2 (let r_i denote the radius of C_i and let $r=r_1-r_2>0$) and an infinite sequence of subcon-

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† A point set which is closed, connected and connected im kleinen is called a *continuous curve*. In general it may be either bounded or unbounded.

‡ For the case where M is the entire plane and N is bounded, see R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), p. 361.

§ See R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, this Bulletin, vol. 29 (1923), p. 296. This theorem is stated by Moore for a bounded continuum, but the theorem remains true without the condition of boundedness.

tinua of K, K_∞, K_1, K_2, K_3, $\cdots$, such that (1) each of these continua $K_\alpha(\alpha = \infty, 1, 2, \cdots)$ contains a point a_α on C_1 and a point b_α on C_2, but no point exterior to C_1 or interior to C_2, (2) no two of these continua have a point in common and no one of them, except possibly K_∞, is a proper subset of any connected subset of K which contains no point without C_1 or within C_2, (3) the set K_∞ is the sequential limiting set of the sequence of continua K_1, K_2, K_3, $\cdots$. For any i, let $K_i^* = K_\infty + K_{i+1} + K_{i+2} + \cdots$. Let d_1 be the smaller of the numbers $\frac{1}{4}r$ and $\frac{1}{2}d(K_1, K_1^*)$†.

Let R_1 denote the set of all points $[P]$ of N such that the distance from P to some point of K_1 is less than d_1. The set R_1 is an open subset of N and hence R_1 contains an arc from a_1 to b_1.‡ This arc contains a subarc x_1y_1 such that x_1 is on C_2 and y_1 is on C_1 and every other point of x_1y_1 is between C_1 and C_2.

For each $n > 1$, let d_n be the smallest of the numbers d_{n-1}, $r2^{-n-1}$, and $\frac{1}{2}d(K_n, K_n^* + x_1y_1 + x_2y_2 + \cdots + x_{n-1}y_{n-1})$. Let R_n denote the set of all points $[P]$ of N such that there is some point of K_n whose distance from P is less than d_n. The set R_n contains an arc from a_n to b_n and this arc contains a subarc x_ny_n such that x_n is on C_2, y_n is on C_1, and every other point of x_ny_n is between C_1 and C_2.

There exists an increasing sequence of positive integers $n_1, n_2, n_3, \cdots$, such that (1) C_2 contains three points X_1, X_2, X_3 such that every point x_{n_i} lies on the arc $X_1X_2X_3$ of C_2 and in the order $X_1X_2x_{n_1}x_{n_2}\cdots X_3$, (2) C_1 contains three points Y_1, Y_2, Y_3 such that every point y_{n_i} lies on the arc $Y_1Y_2Y_3$ of C_1 and in the order $Y_1Y_2y_{n_1}y_{n_2}\cdots Y_3$, (3) X_3 is the sequential limit point of $[x_{n_i}]$ and Y_3 is the sequential limit point of $[y_{n_i}]$. Clearly K_∞ contains X_3 and Y_3.

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† If A and B are sets of points, the symbol $d(A,B)$ denotes the lower bound of all numbers $d(x,y)$, where x is a point of A, y is a point of B, and $d(x,y)$ is the distance from x to y.

‡ R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), p. 255.

The set K_∞ contains points X and Y on the circles which are concentric with C_1 and with radii $r_2+r/10$ and $r_1-r/10$ respectively. Let η be the smaller of the two numbers $r/10$ and r_2. Since N is connected im kleinen, there exists a positive number δ_η such that any point of N within a distance δ_η of X or Y may be joined to X or Y, as the case may be, by an arc of N every point of which is within a distance η of X or Y. Let n_s be the smallest integer such that $x_{n_s}y_{n_s}$ contains two points P and Q such that $d(P, X)<\delta_\eta$ and $d(Q, Y)<\delta_\eta$. Then N contains arcs PX and QY, every point of which is within a distance η of X and Y respectively. Let order be defined on these arcs as *from P to X* and *from Q to Y*. The arcs PX and QY have points in common with every arc $x_{n_i}y_{n_i}$ for $i \geqq s$.

Let V_{10} and V_{20} be the last points the arcs PX and QY have in common with $x_{n_s}y_{n_s}$. Let U_{11} and U_{21} be the first points and V_{11} and V_{21} be the last points the subarcs $V_{10}X$ and $V_{20}Y$ have in common with $x_{n_{s+1}}y_{n_{s+1}}$. Let U_{12} and U_{22} be the first points the subarcs $V_{11}X$ and $V_{21}Y$ have in common with $x_{n_{s+2}}y_{n_{s+2}}$. The set J, composed of the arcs $V_{10}V_{20}$ of $x_{n_s}y_{n_s}$, $U_{11}V_{11}$ and $U_{21}V_{21}$ of $x_{n_{s+1}}y_{n_{s+1}}$, $U_{12}U_{22}$ of $x_{n_{s+2}}y_{n_{s+2}}$, $V_{10}U_{11}$ and $V_{11}U_{12}$ of PX, $V_{20}U_{21}$ and $V_{21}U_{22}$ of QY, is a simple closed curve. The subarc of $x_{n_{s+1}}y_{n_{s+1}}$ lying within J and the arc $x_{n_{s+2}}y_{n_{s+2}}$ have points p_1 and p_3, respectively, in common with the circle concentric with C_1 and with radius $r_2+\frac{1}{2}r$. The point p_1 is interior to J and within a distance $d_{n_{s+1}}$ of some point of K and this point is a limit point of D. Thus D contains a point in the interior of J. Similarly D contains a point within a distance $d_{n_{s+3}}$ of p_3 and thus in the exterior of J. Since D is connected D must contain a point of J. But as J belongs to N and D to $M-N$, D cannot contain a point of J. Thus the assumption that K is not connected im kleinen has led to a contradiction.

THEOREM II. *If a maximal connected subset K of the boundary of an M-domain is a continuous curve, every closed and connected subset of K is a continuous curve.*

DEFINITION. If D is an M-domain and P is a point of $M-\overline{D}$, the M-boundary of the maximal connected subset of $M-\overline{D}$ containing P will be called the *M-boundary of D with respect to P*. This is a generalization of the notion of outer boundary as defined by R. L. Moore.* If M is the entire plane, D is bounded, and P is a point of the maximal connected subset of $M-\overline{D}$ which is unbounded, then the M-boundary of D with respect to P is exactly the outer boundary of D as defined by Moore.*

THEOREM III.† *If D is an M-domain, P is a point of $M-\overline{D}$, and B is the M-boundary of D with respect to P, then B is the entire M-boundary of some M-domain which contains D.*

PROOF. The entire set D lies in the same maximal connected subset of $M-B$ and let R denote this maximal connected subset. Evidently R is an M-domain containing D. Suppose Q is a point of the M-boundary of R. If Q does not belong to B, Q belongs to a maximal connected subset of $M-B$ which is different from R. Then Q is not a limit point of R.‡ Therefore every point of the M-boundary of R is a point of B. Conversely every point of B is an M-boundary point of D and thus of R. Hence B is identical with the M-boundary of R, an M-domain containing D.

THEOREM IV. *If (1) D is an M-domain and P is a point of $M-\overline{D}$, (2) every maximal connected subset of D' is a continuous curve, (3) the M-boundary of D with respect to P, which we denote by B, is bounded, then every maximal connected subset of B is either a point, a simple continuous arc or a simple closed curve.*

PROOF. Let R be the maximal connected subset of $M-\overline{D}$ containing P, and let B_1 be a maximal connected subset of B.

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* *Concerning continuous curves in the plane*, loc. cit., p. 256.

† Compare R. L. Moore, *Concerning continuous curves in the plane*, loc. cit., Theorem 3, p. 258.

‡ See R. L. Moore, *A characterization of a continuous curve*, Fundamenta Mathematicae, vol. 7 (1925), Lemma 1, p. 302.

By Theorem II, B_1 is a continuous curve and B_1 is bounded by hypothesis. If B_1 consists of a single point, our theorem is proved. If B_1 consists of more than a single point then by a theorem due to Mazurkiewicz,* B_1 contains two points x and y which do not cut B_1. The continuous curve B_1 contains an arc xzy from x to y. If $B_1 \equiv xzy$, our theorem is proved. If not, let p be a point of B_1 which does not lie on xzy. By Theorem III, B is the entire M-boundary of some M-domain H which contains D. Clearly R and H are mutually exclusive and B is the entire M-boundary of each. By a theorem due to Wilder,† if p_1 and p_2 are points of R and H respectively there exist arcs p_1x and p_1y which lie except for x and y wholly in R and arcs p_2x and p_2y which lie except for x and y wholly in H. The sets p_1x+p_1y and p_2x+p_2y contain arcs xuy and xvy which lie wholly in R and H respectively except for the points x and y.

Let J_1, J_2, J_3 be the simple closed curves formed of $xuy+xzy$, $xvy+xzy$, $xuy+xvy$ respectively and let I_i and E_i denote the interior and exterior of $J_i (i=1, 2, 3)$. We have three cases to consider:

Case (1). Suppose $I_3 = I_1 + I_2 + <xzy>$.‡ Any point q of $B-xzy$ lies either in I_1, I_2 or E_3. If q lies in I_1, I_1 contains a point of H since q is a limit point of H. The exterior E_1 contains $<xvy>$ of H. But H is connected and contains no point of J_1. Hence I_1 contains no point of $B-xzy$. Similarly I_2 contains no point of $B-xzy$. Then every point of $B-xzy$ lies in E_3. Since x and y are not cut-points of B_1, the continuous curve B_1 contains an arc px which does not contain y and an arc py which does not contain x. The set $px+py$ contains an arc xwy from x to y. Since p is in E_3 and no point of J_3 except x and y is a point of B_1, the set $<xwy>$ lies

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* *Un théorème sur les lignes de Jordan*, Fundamenta Mathematicae vol. 2 (1921), pp. 119–130.

† Loc. cit., Theorem 1, p. 342.

‡ If xzy denotes a simple continuous arc with end-points x and y $<xzy>$ denotes $xzy-(x+y)$.

entirely in E_3, and thus the arcs xwy and xzy have only x and y in common. Let J_4 be the simple closed curve $xzy+xwy$. We will show that $B-J_4$ is vacuous and thus prove $B\equiv B_1\equiv J_4$. Suppose $B-J_4$ contains a point q_1. If q_1 lies in the interior of J_4 both R and H have points in the interior of J_4 since q_1 is a limit point of both domains. One of the two sets $<xuy>$ or $<xvy>$, say $<xuy>$, lies entirely in the exterior of J_4. Then R contains points interior and exterior to J_4 but contains no point of J_4, which is impossible. If q_1 lies in the exterior of J_4, both R and H contain points in the exterior of J_4, and one of them contains points in the interior, which is impossible. Therefore $B-J_4$ is vacuous, which proves the theorem for this case.

Case (2). Suppose $I_2=I_1+I_3+<xuy>$.

Case (3). Suppose $I_1=I_2+I_3+<xvy>$.

In Cases (2) and (3), it may be proved by methods similar to those of Case (1) that $B\equiv B_1$ and B_1 is a simple closed curve. Therefore B_1 is either a point, a simple continuous arc, or a simple closed curve.

In proving Theorem IV we have obtained this result:

THEOREM V. *Under the hypothesis of Theorem* IV, *if any maximal connected subset J of the M-boundary of D with respect to P is a simple closed curve, then J is the entire M-boundary of D with respect to P.*

THEOREM VI. *If D is an M-domain, P is a point of $M-\overline{D}$, R is the maximal connected subset of $M-\overline{D}$ containing P, and Q is a point of the maximal connected subset of $M-\overline{R}$ which contains D, then R' is the M-boundary of R with respect to Q.*

PROOF. Let H denote the maximal connected subset of $M-\overline{R}$ which contans D. By definition H' is the M-boundary of R with respect to Q. The set H' is a subset of R' since the M-boundary of a domain with respect to a point is always a subset of the M-boundary of the domain. By definition, R' is the M-boundary of D with respect to P. Then every point of R' is a limit point of D and thus of H. As H is a subset of

$M-\overline{R}$, no point of R' is a point of H. Therefore every point of R' is an M-boundary point of H, that is, R' is a subset of H'. Hence $R' \equiv H'$.

THE UNIVERSITY OF PENNSYLVANIA

A THEOREM ON CONNECTED POINT SETS

BY C. KURATOWSKI AND C. ZARANKIEWICZ

1. *Introduction.* The purpose of this paper is to prove the following theorem.

THEOREM. *If S is a connected point set and Z is the set of all points such that $S-p$ is neither connected nor the sum of two connected sets, then Z is finite or countable.*

2. LEMMA. *If S, P, and Q are three non-vacuous connected sets (or points), and if*

(1) $P+Q \subset S$, (2) $P \cdot Q = 0$,

(3) $A \subset S-P$, (4) $B \subset S-Q$,

(5) $A \cdot Q = 0$, (6) $B \cdot P = 0$,

(7) *A and $S-P-A$ are mutually separated,*

(8) *B and $S-Q-B$ are mutually separated,*

then $A \cdot B = 0$.

PROOF. By (1) and (3), $A+P \subset S$. Hence, by (2) and (5), $A+P=(A+P)\cdot(S-Q)$. By (4), $S-Q=B+(S-Q-B)$. Therefore

$$(9) \qquad A+P=(A+P)\cdot B+(A+P)\cdot(S-Q-B).$$

It follows from (2) and (6) that $P=P-Q-B \subset (A+P)\cdot(S-Q-B)$. Since $P \neq 0$, we have

$$(10) \qquad (A+P)\cdot(S-Q-B) \neq 0.$$

Now, by (8), the sets $(A+P)\cdot B$ and $(A+P)\cdot(S-Q-B)$ are mutually separated. On the other hand, by virtue of a

theorem of Knaster and Kuratowski,* and by (3) and (7), the set $A+P$ is connected. Hence, by (9) and (10), $(A+P)\cdot B=0$, that is, $A\cdot B=0$.

3. *Proof of the Theorem.* Let

$$R_1, R_2, \cdots, R_n, \cdots \tag{11}$$

be the sequence of all circles (spheres) which are such that their radii and the coordinates of the center are rational; and let

$$(1,2,3),(1,2,4), \cdots ,(i,j,k), \cdots \tag{12}$$

be the sequence of all systems composed of three different positive integers.

We shall first show that, for every point p of Z, there exist three sets A, C, and D and a system of indices (i, j, k) such that

(13) *A, C, and D are mutually separated,*

$$S - p = A + C + D, \tag{14}$$

$$S\cdot R_i = A\cdot R_i \neq 0, \tag{15}$$

$$S\cdot R_j = C\cdot R_j \neq 0 \neq D\cdot R_k = S\cdot R_k. \tag{16}$$

By hypothesis, $S-p$ is not connected. Hence $S-p$ may be decomposed into two mutually separated sets; one of these two sets can itself be decomposed into two mutually separated sets, since by hypothesis $S-p$ is not the sum of two connected sets. It follows that there exists at least one decomposition of $S-p$ into three non-vacuous mutually separated sets A, C, and D. Let a, c, and d be three points in A, C, and D, respectively. It is obvious that the point a may be surrounded by a circle R_i which contains no point of $C+D+p$. Similarly c and d belong to two circles R_j and R_k that contain no point of $A+D+p$ and of $A+C+p$, respectively. Therefore the conditions (15) and (16) are fulfilled.

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* *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2 (1921), p. 210, Theorem VI.

Thus we have shown the existence of a system (i, j, k) having the property that there exists at least one system of three sets (A, C, D) satisfying the formulas (13)–(16). We denote by $\sigma(p)$ the *first* system in the sequence (12) having that property.

We shall prove that, if

$$p \neq q, \tag{17}$$

then $\sigma(p) \neq \sigma(q)$.

Suppose, on the contrary, that $\sigma(p) = \sigma(q) = (i, j, k)$. Hence there exist three sets A, C, D satisfying the conditions (13)–(16) and three sets B, E, F such that

(18) *$B, E,$ and F are mutually separated,*

$$S - q = B + E + F, \tag{19}$$

$$S \cdot R_i = B \cdot R_i \neq 0, \tag{20}$$

$$S \cdot R_j = E \cdot R_j \neq 0 \neq F \cdot R_k = S \cdot R_k. \tag{21}$$

From (15) and (20) it follows that $A \cdot R_i = B \cdot R_i \neq 0$. Hence

$$A \cdot B \neq 0 \tag{22}$$

and similarly, $C \cdot E \neq 0 \neq D \cdot F$.

By (17), (14) and (13) q belongs to one and only one of the sets A, C, and D. Similarly, p belongs to one and only one of the sets B, E, and F. Clearly we may suppose that neither q belongs to A nor p to B, so that

$$A \cdot q = 0 = B \cdot p. \tag{23}$$

Now, by (13) and (14), $C + D = S - p - A$. It follows by (13) that A and $S - p - A$ are mutually separated. Similarly, B and $S - q - B$ are mutually separated. Therefore, by virtue of the Lemma and by (17), (14), (19), and (23) we have $A \cdot B = 0$. But this is a contradiction of (22).

Thus we have shown that to different points of the set Z correspond different systems composed of three positive integers. Hence the set Z is countable.

4. *Remarks.* A. It is to be noted that the above proof was carried out *without the aid of the Zermelo Axiom** and that it gives a mode of enumerating the points of Z. That is, Z is *effectively countable* (in the sense of Borel-Sierpinski).

B. By means of a remark of B. Knaster, the argument used above may be applied to prove the following more, general theorem. *If S is a connected set and Z is a class of mutually exclusive, connected, and relatively (to S) closed sets X, such that $S-X$ is neither connected nor the sum of two connected sets, then the class Z is countable.*

C. In case S is an acyclic continuous curve (i. e., a continuous curve that contains no simple closed curves), then the points of the set Z are identical with *points of ramification* of S.† Thus, our Theorem has as a consequence the theorem of Wazewski-Menger‡ to the effect that *the set of points of ramification of a tree is countable.*

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* The Zermelo Axiom was used by Wazewski (Annales de la Société Polonaise de Mathématique, vol. 2 (1923), p. 169) to prove a particular case of our Theorem, namely the case S is a continuous curve.

By a method analogous to that used above, the following theorem that was proved with the aid of the Zermelo Axiom by R. L. Moore in a less general form (see Proceedings of the National Academy of Sciences, vol. 9 (1923), p. 102), and by C. Zarankiewicz (Fundamenta Mathematicae, vol. 9 (1927), p. 140, Theorem 9) may be proved without having reference to that axiom:

If T is a connected subset of a connected set S and W is the set of points each of which belongs to T and disconnects S but not T, then W is countable.

For, suppose p is a point of W. Then there exists a circle R having the property that there exists a set A such that (i) A and $A-p-S$ are mutually separated, (ii) $A \subset S-T$, (iii) $S \cdot R = A \cdot R \neq 0$.

Let $n(p)$ denote the lowest index such that the circle $R_{n(p)}$ of the sequence (11) has that property. Now, if $p \neq q$ then $n(p) \neq n(q)$. Suppose, on the contrary, that $n(p) = n(q)$. Hence there exists a set B such that (iv) B and $S-q-B$ are mutually separated, (v) $B \subset S-T$, (vi) $S \cdot R = B \cdot R \neq 0$. It follows, by (ii), that $A \cdot q = 0$ and, by (v), that $B \cdot p = 0$. Therefore, by (i), (iv), and the Lemma, we have $A \cdot B = 0$. But this is a contradiction of (iii) and (vi). Thus the set W is effectively countable.

† See K. Menger, *Ueber reguläre Baumkurven*, Mathematische Annalen, vol. 96 (1926), p. 574. A point p is said to be a *point of ramification* of a (regular) curve if the curve contains three arcs L_1, L_2, L_3 such that $p = L_1 \cdot L_2 = L_1 \cdot L_3 = L_2 \cdot L_3$.

‡ Wazewski, loc. cit., p. 169, Menger, loc. cit., p. 576.

This theorem may be generalized as follows. Suppose S is a acyclic continuous curve *locally* at each of its points (i. e., for each point p of S there exists an acyclic continuous curve T_p contained in S and such that p is not a limit point of $S-T_p$). It follows, by the Borel covering theorem, that the set of all points of ramification of S is still countable.

A direct consequence of this statement is the following result* obtained by Alexandroff: *If S is a continuous curve of finite connectivity† then the set of all points of ramification of S is countable.* For,‡ a continuous curve of finite connectivity is a tree locally at each of its points.

Now, if S is a plane non-dense continuous curve that disconnects the plane in a finite number of domains, then§ S has a finite connectivity. Thus we may state the following result. *The set of points of ramification of a plane non-dense continuous curve that disconnects the plane in a finite number of domains is countable.*‖

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* P. Alexandroff, *Ueber kombinatorische Eigenschaften allgemeiner Kurven*, Mathematische Annalen, vol. 96 (1926), p. 552, Corollary 4.

† A (one-dimensional bounded) continuous curve is said to be of *finite connectivity* if it contains at most a finite number of simple closed curves. This definition is equivalent to that given by Alexandroff, loc. cit., p. 541, by virtue of the Lemma 15 of the mentioned paper of Zarankiewicz.

‡ See Menger, loc. cit., p. 574.

§ See Alexandroff, loc. cit., p. 527.

‖ This statement does not remain true if the hypothesis that the number of complementary domains is *finite* be omitted, as is seen on an example of a continuous (regular) curve composed exclusively of points of ramification, given by W. Sierpinski, Comptes Rendus, vol. 160 (1915), p. 305; see also this Bulletin, vol. 33 (1927), p. 106.

ON CERTAIN SUFFICIENT CONDITIONS FOR THE CONVERGENCE AND CESÀRO SUMMABILITY OF THE ALLIED SERIES OF A DOUBLE FOURIER SERIES*

BY G. M. MERRIMAN†

1. *Introduction.* The double Fourier series of a summable function

$$f(\alpha,\beta) \sim \sum_1^\infty \sum_1^\infty \left\{ \begin{matrix} a_{i,j} \cos i\alpha \cos j\beta + b_{i,j} \cos i\alpha \sin j\beta \\ + c_{i,j} \sin i\alpha \cos j\beta + d_{i,j} \sin i\alpha \sin j\beta \end{matrix} \right\} \tag{1}$$

$$= \sum_1^\infty \sum_1^\infty (a,b,c,d,\alpha,\beta)_{i,j}$$

has three series said to be allied to it :

$$\text{(A)} \qquad \sum_1^\infty \sum_1^\infty (d, -c, -b, a, \alpha, \beta)_{i,j},$$

$$\text{(B)} \qquad \sum_1^\infty \sum_1^\infty (c, d, -a, -b, \alpha, \beta)_{i,j},$$

$$\text{(C)} \qquad \sum_1^\infty \sum_1^\infty (b, -a, d, -c, \alpha, \beta)_{i,j}.$$

These will be designated hereafter by the terms first, second, and third allied series, respectively.

Although the theorems concerning these series to be stated in this paper follow quite readily, in fact, so readily that most of the details of proof will be omitted, from the use of certain auxiliary functions and identities, yet they are useful and do not appear elsewhere in the literature. They were suggested by three papers of W. H. Young.‡

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† National Research Fellow.

‡ (1) *Multiple Fourier series*, Proceedings of the London Society, (2), vol. 11 (1912–1913); (2) *On the convergence of a Fourier series and its allied series*, Proceedings of the London Society, (2), vol. 10 (1911–1912); (3)

2. *Convergence Theorems.* If we write

$$\phi_1(x,y) \equiv \frac{1}{4}\{f(\alpha+x,\beta+y) - f(\alpha+x,\beta-y) - f(\alpha-x,\beta+y) + f(\alpha-x,\beta-y)\}$$
$$\sim \sum_1^\infty \sum_1^\infty (d, -c, -b, a, \alpha, \beta)_{i,j} \sin ix \sin jy,$$

$$\phi_2(x,y) \equiv \frac{1}{4}\{f(\alpha+x,\beta+y) + f(\alpha+x,\beta-y) - f(\alpha-x,\beta+y) - f(\alpha-x,\beta-y)\}$$
$$\sim \sum_1^\infty \sum_1^\infty (c, d, -a, -b, \alpha, \beta)_{i,j} \sin ix \cos jy,$$

$$\phi_3(x,y) \equiv \frac{1}{4}\{f(\alpha+x,\beta+y) - f(\alpha+x,\beta-y) + f(\alpha-x,\beta+y) - f(\alpha-x,\beta-y)\}$$
$$\sim \sum_1^\infty \sum_1^\infty (b, -a, d, -c, \alpha, \beta)_{i,j} \cos ix \sin jy,$$

we may state the following theorem.

THEOREM I. *If $f(\alpha, \beta)$ is a function of bounded variation*, and if each of the integrals*

$$(2) \qquad \lim_{\epsilon_1,\epsilon_2 \to 0} \frac{1}{\pi^2} \int_{\epsilon_1}^{\pi} \int_{\epsilon_2}^{\pi} \phi_1(x,y) \operatorname{ctn}(x/2) \operatorname{ctn}(y/2)\, dx\, dy,$$

$$(3) \qquad \lim_{\epsilon_1 \to 0} \frac{1}{\pi^2} \int_{\epsilon_1}^{\pi} \int_{0}^{\pi} \phi_2(x,y) \operatorname{ctn}(x/2)\, dx\, dy,$$

$$(4) \qquad \lim_{\epsilon_2 \to 0} \frac{1}{\pi^2} \int_{0}^{\pi} \int_{\epsilon_2}^{\pi} \phi_3(x,y) \operatorname{ctn}(y/2)\, dx\, dy,$$

Konvergenzbedingungen für die verwandte Reihen einer Fouriersche Reihe, Akademie der Wissenschaften München, 1911. (1) will hereafter be referred to as M.F.S., (2) as C.A.S., (3) as K.V.R.

* See M.F.S., §3; a function of two variables is said to have bounded variation if it is of bounded variation not only with respect to both variables, but also with respect to each variable separately.

has a finite limit, then the three allied series of $f(\alpha, \beta)$ converge, (A) *to* (2), (B) *and* (C) *to the negatives of* (3) *and* (4), *respectively.*

Theorem I is a generalization of the result in K.V.R., §2. By slight changes in the details of proof, we may obtain also two more general theorems, which will be needed later.

THEOREM II. *If $\psi(u, v)$ is an odd-odd function of u and v, and has the property that $uv\psi(u, v)$is the double integral of some function $\phi(u, v)$ such that*

$$\int_0^u \int_0^v |\phi(u,v)|\, du\, dv = o(uv),$$

then the series of double Fourier constants of $\psi(u, v)$ (i.e., the first allied series of $\psi(u, v)$ at $u=0, v=0$) converges to the value

$$\lim_{\epsilon_1,\epsilon_2\to 0} \frac{1}{\pi^2} \int_{\epsilon_1}^{\pi} \int_{\epsilon_2}^{\pi} \psi(u,v)\ \mathrm{ctn}\ (u/2)\ \mathrm{ctn}\ (v/2)\, du\, dv,$$

provided that this limit exists.

THEOREM III. *If $\psi(u, v)$ is an odd-even function of u and v, such that $u\psi(u, v)$ is the integral with respect to u of a function $\phi(u, v)$ such that*

$$\int_0^u |\phi(u,v)|\, du = o(u),$$

and $\phi(u, v)$ is, near $u=0, v=0$, a continuous function of v, and a bounded function of u and v, then the series of double Fourier constants of $\psi(u, v)$ converges to the limit

$$\lim_{\epsilon_1\to 0} \frac{-1}{\pi^2} \int_{\epsilon_1}^{\pi} \int_0^{\pi} \psi(u,v)\ \mathrm{ctn}\ (u/2)\, du\, dv,$$

*if it exists.**

A theorem similar to Theorem III may be stated for an even-odd function.

* The continuity of $\psi(u,v)$ at the origin, essential to the proof of this theorem, follows from a result of H. J. Ettlinger: *On continuity in several variables*, this Bulletin, vol. 33, No. 1 (Jan., 1927).

3. *Auxiliary Identities and Theorems.* We write

$$g_1(u,v) = \csc u \csc v \int_0^u \int_0^v \phi_1(2u,2v)\, du\, dv,$$

$$g_2(u,v) = \csc u \int_0^u \phi_2(2u,2v)\, du,$$

$$g_3(u,v) = \csc v \int_0^v \phi_3(2u,2v)\, dv.$$

If we carry out reductions akin to those in M.F.S., §§13, 14, and 21, we derive the following set of identities:

(I) $s_{m,n} = S_{m,n} - p_{m,n} - q_{m,n} + mnA_{m,n}$,

(II) $[s_{C,C}]_{m,n} = 4[S_{C,C}]_{m,n} - 2[S_{C,O}]_{m,n} - 2[S_{O,C}]_{m,n} + S_{m,n}$,

(III) $[s_{C,C}]_{m,n} = 2[S_{C,C}]_{m,n} - [S_{O,C}]_{m,n}$,

(IV) $[s_{C,C}]_{m,n} = 2[S_{C,C}]_{m,n} - [S_{C,O}]_{m,n}$.

In (I), $s_{m,n}$ stands for the partial sum of (A), $S_{m,n}$ for the partial sum of the double Fourier constants of $g_1(u, v)$, $p_{m,n}$ for m times the sum of the first n terms of the mth column of the double Fourier constants of $g_1(u, v)$, $q_{m,n}$ for n times the sum of the first m terms of the nth row, and $A_{m,n}$ is the general term of the double Fourier constants of $g_1(u, v)$. In (II), the $s_{m,n}$ and $S_{m,n}$ stand for the same partial sums as in (I) and the subscripts C and O stand for "Cesàro" and "ordinary" summations respectively. In (III) and (IV), the $s_{m,n}$ is the partial sum of (B) and (C), the $S_{m,n}$ the partial sum of the double Fourier constants of $g_2(u, v)$ and $g_3(u, v)$, respectively. The last three identities can be extended to suitably defined Cesàro sums of higher indices. With these identities at hand, we may state the following results.

From repeated applications of (I) and (II), we obtain

THEOREM IV. *If $g_1(u, v)$ be a summable function of u and v such that the functions $p_{m,n}$, $q_{m,n}$ and $mnA_{m,n}$ of its double Fourier constants have the unique double limit zero, then, if it be known that either the first allied series of $f(\alpha, \beta)$ or that of*

$g_1(u, v)$, at $u=0, v=0$, converges doubly in any Cesàro manner, indices (r, r), it follows that both series converge doubly in the ordinary manner and have the same sum.*

COROLLARY. *If the functions $p_{m,n}$, $q_{m,n}$ and $mnA_{m,n}$ converge to limits other than zero, the theorem is still true, except that the series then have not the same sum; the difference in their sums is the sum of the limits of the functions.*

If we use (II) alone, and its extensions noted above, we have the following theorem.

THEOREM V. *If the first allied series of the double Fourier series of $g_1(u, v)$ converges doubly in the ordinary way at the point $u=0, v=0$, then the first allied series of $f(\alpha, \beta)$ converges doubly in the Cesàro manner, indices (1, 1), and the sums of the series are the same.*

This theorem can be extended to general Cesâro indices, and the footnote appended to Theorem IV shows that they need not be the same.

If we use (III) in the same manner, we obtain the theorem

THEOREM VI. *If the second allied series of $g_2(u, v)$ converges doubly in the ordinary way at the point $u=0$, $v=0$, then the second allied series of $f(\alpha, \beta)$ converges $(C, 1, 1)$, and the sums of the series are the same.*

By use of (IV), a theorem similar to Theorem VI can be stated relative to the third allied series. Also, the theorems can be extended to higher Cesâro indices. Of course, we have not used the narrowest possible hypothesis by requiring the double convergence of the allied series of $g_2(u, v)$; we might have arrived at the same conclusions by requiring only the ordinary-Cesâro convergence. However, we shall need the result only as here stated.

We must note finally that we can replace the functions $g_1(u, v)$, $g_2(u, v)$, and $g_3(u, v)$ by other auxiliary functions

* That these indices of summation may be different, say (r, s), follows as in M.F.S., §§17, 18.

$h_1(u, v)$, $h_2(u, v)$, and $h_3(u, v)$, in which the factors csc u and csc v have been replaced by $1/u$ and $1/v$ respectively.*

4. *Summability* $(C, 1, 1)$. We proceed now to the main theorems of the paper.

THEOREM VII. *If*

$$\int_0^u \int_0^v |\phi_1(2u, 2v)| \, du \, dv = o(uv),$$

and the limit

$$\lim_{\epsilon_1, \epsilon_2 \to 0} \frac{1}{\pi^2} \int_{\epsilon_1}^{\pi} \int_{\epsilon_2}^{\pi} \phi_1(u, v) \operatorname{ctn} (u/2) \operatorname{ctn} (v/2) \, du \, dv \tag{5}$$

exists, then the first allied series of $f(\alpha, \beta)$ *sums* $(C, 1, 1)$ *to* (5).

The function $\phi_1(2u, 2v)$ fulfils the rôle of $\phi(u, v)$ in Theorem II, and therefore the first allied series of $h_1(u, v)$, and hence that of $g_1(u, v)$, converges at $u=0$, $v=0$, to the limit (if it exists)

$$\lim_{\epsilon_1, \epsilon_2 \to 0} \frac{1}{\pi^2} \int_{\epsilon_1}^{\pi} \int_{\epsilon_2}^{\pi} g_1(u, v) \operatorname{ctn} (u/2) \operatorname{ctn} (v/2) \, du \, dv.$$

Therefore, by Theorem V, the first allied series of $f(\alpha, \beta)$ sums $(C, 1, 1)$ to the value, if it exists,

$$\lim_{\epsilon_1, \epsilon_2 \to 0} \frac{1}{4\pi^2} \int_{\epsilon_1}^{\pi} \int_{\epsilon_2}^{\pi} \left\{ \int_0^u \int_0^v \phi_1(2u, 2v) du \, dv \right\} \cdot \csc^2(u/2) \csc^2(v/2) du \, dv. \tag{6}$$

Integrating this result doubly by parts and using hypothesis, we complete Theorem VII.

COROLLARY. *The limit* (5) *and the limit*

$$\lim_{\epsilon_1, \epsilon_2 \to 0} \frac{1}{\pi^2} \int_{\epsilon_1}^{\infty} \int_{\epsilon_2}^{\infty} \frac{\phi_1(u, v)}{uv} du \, dv \tag{7}$$

are interchangeable, provided the periodicity of $f(\alpha, \beta)$ *is used to extend its definition outside the given range.*

* See C.A.S., §§5 and 12.

The expression (6) can be written in the form

$$\lim_{\epsilon_1,\epsilon_2\to 0}\frac{4}{\pi^2}\int_{\epsilon_1}^{\infty}\int_{\epsilon_2}^{\infty}u^{-2}v^{-2}\left\{\int_0^u\int_0^v\phi_1(2u,2v)du\,dv\right\}du\,dv,$$

whence, integrating doubly by parts and using the hypothesis, the corollary is proved.

THEOREM VIII. *If*

$$\int_0^u\int_0^v|\phi_2(2u,2v)|du\,dv=o(uv),$$

and the limit

$$\lim_{\epsilon_1\to 0}\frac{-1}{\pi^2}\int_{\epsilon_1}^{\pi}\int_0^{\pi}\phi_2(u,v)\operatorname{ctn}(u/2)\,du\,dv \tag{8}$$

exists, then the second allied series of $f(\alpha,\beta)$ *sums* $(C,1,1)$ *to* (8).

The function $\phi_2(2u, 2v)$ is easily seen to play the part of $\phi(u, v)$ in Theorem III, and hence the second allied series of $h_2(u, v)$ and therefore that of $g_2(u, v)$ converge to the limit (if it exists)

$$\lim_{\epsilon_1\to 0}\frac{-1}{\pi^2}\int_{\epsilon_1}^{\pi}\int_0^{\pi}g_2(u,v)\operatorname{ctn}(u/2)\,du\,dv.$$

Hence, by Theorem VI, the second allied series of $f(\alpha, \beta)$ sums $(C, 1, 1)$ to

$$\lim_{\epsilon_1\to 0}\frac{-1}{2\pi^2}\int_{\epsilon_1}^{\pi}\int_0^{\pi}\left\{\int_0^u\phi_2(2u,2v)du\right\}\csc^2(u/2)du\,dv. \tag{9}$$

An integration by parts with respect to u, and the use of hypothesis, complete Theorem VIII.

A theorem similar to Theorem VIII can be deduced concerning the third allied series of $f(\alpha, \beta)$.

In conclusion, mention should, be made perhaps, of theorems analogous to those of §§3 and 4, dealing with summability $(C, r, 0)$ and $(C, 0, s)$ of the allied series, which can be developed along the lines of this paper from functions

of the type of $g_2(u, v)$ and $g_3(u, v)$. It would be such theorems which would allow us to arrive at the statements contained in the footnote appended to Theorem IV and elsewhere.

In the proofs of Theorems V and VI, use was made of a "consistency theorem" to the effect that if a double series is summable (C, r, s), it is also summable (C, r', s'), $r' \geqq r, s' \geqq s$, and to the same sum as formerly. This result is contained in Theorem I′ of a paper by the author concerning the summability of double series of a certain type, which will appear in an early issue of the Annals of Mathematics.

HARVARD UNIVERSITY

A GENERAL THEOREM ON QUANTIC DETERMINANTS*

BY T. R. ROSEBRUGH

Let there be a homogeneous linear substitution

$$x_{rj}' = \sum_{i=1}^{n_r} a_{rji} x_{ri}, \qquad (j = 1, 2, \cdots, n_r),$$

having M_r for the determinant of its coefficients, and let there be derived therefrom quantics of degree p_r giving values of all possible monomials in these n_r new variables of that degree. There will be

$$f_r = \binom{p_r + n_r - 1}{p_r}$$

such new monomials, and an equal number of the old will be involved in these quantics.

Now consider this to be but one, namely the rth of a set of s such substitutions, each in its own set of variables of arbitrary (generally different) number n_r, similarly treated. Each new monomial of each of the s sets is now to be used,

* Presented to the Society, February 26, 1927.

one only from each set, as a factor in a continued product which contains such factors.

It is now proposed to examine a direct method for finding the algebraic composition of the determinant M of the coefficients of the quantics so constructed, when placed in an array similarly ordered in the composition of the monomials of the new and old quantities. It is evident that this condition of similar ordering is sufficient to ensure the same value of M for any change in the order, since it effects the same permutation of the rows as of the columns.

The result of this examination will be found to be that

$$\log M = \sum_{r=1}^{s} u_r \log M_r$$

where

$$u_r = \frac{p_r}{n_r} \prod_{r=1}^{s} f_r.$$

It is desirable to agree first for descriptive purposes on some rule of order more definite than the mere essential condition of similar order. Let the factor-groups of new variables from all other substitutions than the rth, combined, appearing associated with factor-groups from the rth on the left side of the equations be assigned an arbitrary order which is likewise applied to the corresponding factor-groups of old variables on the right.

There will then be

$$q_r = \frac{1}{f_r} \prod_{r=1}^{s} f_r$$

such factor-groups. Now let the value be supposed written out of the product of each of the f_r primed monomials of the rth group (taken in an assigned order) with the first of the other factor-groups; then the products of each of them in the same order with the second of the q_r other factor-groups, and so on. For convenience each set of f_r rows of the *coefficients* so obtained may be called a block. There will thus be q_r blocks.

The order of the columns of coefficients is fixed by that of the associated monomials of the old variables and is determined by application of the rule of similar order. Thus there will be blocks running horizontally as well as vertically in each of which the same order for rth substitution variables will apply. For convenience, let anything referring to variables or coefficients connected with the rth substitution be called "rth stage." Considering now details of the rth stage coefficients within individual blocks horizontally and vertically, and the associated variables of that stage, and ignoring the factors from the other stages which are present, we proceed as follows:

There is a row in each block giving the value of a new term in which the factor

$$\prod_{j=1}^{n_r} (x'_{rj})^{t'_{rj}}$$

occurs. The ordinal number of this row in the block has been fixed in an assigned manner by the exponents of the system $(t'_{r1}, t'_{r2}, \cdots, t'_{rn_r})$, admitting zero values. The column in which the coefficient of the old term in which the factor

$$\prod_{j=1}^{n_r} (x_{rj})^{t_{rj}}$$

occurs has accordingly been fixed by the same rule of order by the exponents of the system $(t_{r1}, t_{r2}, \cdots, t_{rn_r})$ also admitting zero values.

Suppose now we examine the effect of the operator

$$\sum_{i=1}^{n_r} a_{rli} \frac{\partial}{\partial a_{rji}}$$

on M, the determinant under study, in case (1), $l=j$, and in case (2), $l \neq j$. If a determinant some of whose elements are functions of a variable be differentiated with regard to that variable the result is the sum of the products of the derivative of each element into the corresponding unchanged co-factor. Consequently an operator of the above form acting on a

determinant gives the sum of the products of the result of its application to each element into the corresponding unchanged co-factor. Now these products may be grouped by rows, each row of the results of operation being associated with all the other rows unchanged, and thus a total of as many determinants as the order of the determinant may be so formed.

First consider, however,

$$\sum_{i=1}^{n_r} a_{rli} \frac{\partial}{\partial a_{rji}}$$

applied not to an individual element of the determinant M but to any one of the quantics in all the variables, containing, together with factors from all the other stages, the rth stage factor

$$\prod_{j=1}^{n_r} (x'_{rj})^{t'_{rj}}.$$

This quantic may be written, for a chosen value of j,

$$C(x'_{rj})^{t'_{rj}}$$

where C contains all factors independent of $a_{rji}(i=1, 2, \cdots, n_r)$ partly from its own stage and partly from the aggregate of the others. This is therefore

$$C\left\{ \sum_{i=1}^{n_r} a_{rji}x_{ri} \right\}^{t'_{rj}}.$$

Now when $t'_{rj} \neq 0$, we have

$$\sum_{i=1}^{n_r} a_{rli} \frac{\partial}{\partial a_{rji}} \cdot C \left\{ \sum_{i=1}^{n_r} a_{rji}x_{ri} \right\}^{t'_{rj}}$$

$$= t'_{rj} C \left\{ \sum_{i=1}^{n_r} a_{rli}x_{ri} \right\} \left\{ \sum_{i=1}^{n_r} a_{rji}x_{ri} \right\}^{t'_{rj}-1}.$$

Thus in case (1), $(l=j)$, this quantic is converted into t'_{rj} times itself by the operation (as in Euler's Theorem), while in case (2), $(l \neq j)$, it is converted into t'_{rj} times the

quantic of another row, identifiable by the system of exponents in which t'_{rl} and t'_{rj} are respectively replaced by $t'_{rl}+1$ and $t'_{rj}-1$. The case $t'_{rj}=0$ requires special consideration. In this case the a_{rji} not being present, the result of the operation both for $l=j$ and $l\neq j$ is zero. Hence *every coefficient* in a given row is, for $t'_{rj}=0$, in both cases converted into zero, and for $t'_{rj}\neq 0$, is converted either into t'_{rj} times itself, case (1): or, case (2), into t'_{rj} times the coefficient in the same column in another row identifiable as described above.

In case (2), since the operation yields either zero for each coefficient or else a constant multiple of the corresponding coefficient of another row, the value of each determinant so produced will be zero. That is

$$(1) \qquad \sum_{i=1}^{n_r} a_{rli}\frac{\partial M}{\partial a_{rji}} = 0, \quad (l,j = 1,2, \cdots ,n_r),(l \neq j).$$

This denotes $n_r^2 - n_r$ equations. But in case (1),

$$\sum_{i=1}^{n_r} a_{rli}\frac{\partial M}{\partial a_{rji}}$$

includes for every row a determinant having the value $t'_{rj}\ M$, the case $t'_{rj}=0$ being now included. Now the set of values of t'_{rj} for the individual rows in a block are the same for each block. The total contribution from any one block of rows is $M\sum t'_{rj}$ where it is easily seen that $n_r\sum t'_{rj}=f_r p_r$.

The contribution from one block of rows is thus $f_r p_r M/n_r$ and from all the blocks, which are

$$\frac{1}{f_r}\prod_{r=1}^{s} f_r$$

in number, it is

$$\frac{p_r}{n_r}M\prod_{r=1}^{s} f_r = u_r M,$$

say. Hence

$$(2) \qquad \sum_{i=1}^{n_r} a_{rji}\frac{\partial M}{\partial a_{rji}} = u_r M, \quad (j = 1,2, \cdots ,n_r);$$

that is, n_r equations. Now let (1) and (2) be combined in a single statement thus:

$$(3)\quad \sum_{i=1}^{n_r} a_{rli}\frac{\partial(u_r^{-1}\log M)}{\partial a_{rji}} = \begin{cases} 1, l = j, \\ 0, l \neq j, \end{cases} (l,j = 1,2,\cdots,n_r),$$

and compared with

$$(4)\quad \sum_{i=1}^{n_r} a_{rli}\frac{\partial \log M_r}{\partial a_{rji}} = \begin{cases} 1, l = j, \\ 0, l \neq j, \end{cases} (l,j = 1,2,\cdots,n_r),$$

which describes known properties of a simple determinant.

It is seen at once that the unique solution for the n_r^2 unknown partial derivatives in (3) is

$$\frac{\partial \log M}{\partial a_{rji}} = u_r\frac{\partial \log M_r}{\partial a_{rji}}, \quad (i,j = 1,2,\cdots,n_r),$$

that is

$$(5)\qquad \frac{\partial (\log M - u_r \log M_r)}{\partial a_{rji}} = 0.$$

That is, the n_r^2 quantities a_{rji} occur in $\log M$ in such a manner that when $u_r \log M_r$ involving these n_r^2 quantities only has been subtracted from it they have all disappeared from it, and the remainder consists therefore only of an unaltered function of the coefficients of the other stages.

Taking account also of the fact that the simplest possible substitutions corresponding to unity along all principal diagonals and zero elsewhere would give a similar form to the determinant yielding M, and thus give $M_r = 1$, $(r = 1, 2, \cdots, s)$ and $M = 1$, we conclude that

$$\log M = \sum_{r=1}^{s} u_r \log M_r,$$

where

$$(6)\qquad u_r = \frac{p_r}{n_r}\prod_{r=1}^{s}\binom{n_r + p_r - 1}{p_r}.$$

From this general result others less general may of course be obtained by successive specialization. For this purpose the use of a notation will be convenient. Let

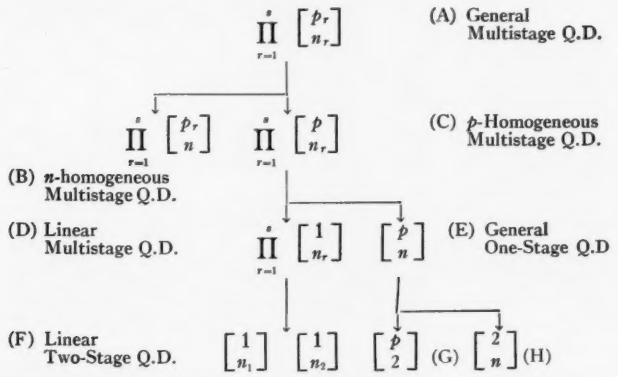
$$\begin{bmatrix} p \\ n \end{bmatrix}$$

denote the theorem relating to the determinant of the coefficients of the quantics representing all monomials of degree p in the n variables given by n equations linearly in terms of others. Then

$$\Pi \begin{bmatrix} p_r \\ n_r \end{bmatrix}$$

may be understood to denote that for the case where all the monomials formed as products from different substitutions are expressed as quantics in variables from the other sets, namely the general theorem proved above.

The relationship between the general theorem and others derivable as special cases may be expressed in the following diagram, in which the abbreviation Q.D. is used for Quantic Determinant.



(G) = Binary One-Stage Quantic Determinant.
(H) = Quadratic One-Stage Quantic Determinant.

Of the above, the only cases known to the writer as having been previously studied are those designated (F), (G) and (H) as follows:

(F): E. Pascal, *I Determinanti*, Milan, 1897, page 138, names this theorem "Teorema di Kronecker," but the reference which he gives is incorrect. The exponents are $u_1 = n_2$ and $u_2 = n_1$.

(G): A theorem of Faà di Bruno deals with this case. The demonstration is given on pages 228–29 of his *Théorie des Formes Binaires*, Turin, 1876. The exponent u of the original determinant here becomes $p(p+1)/2$.

(H): These determinants are discussed by Hunyady, Journal für Mathematik (vol. 89 (1879), p. 58). H. Leitzmann, in his translation of Pascal's *I Determinanti*, credits the theorem to Scholtz in place of Hunyady, as in the original. The exponent u proves to be $n+1$.

THE UNIVERSITY OF TORONTO

ON THE DEGREE OF APPROXIMATION TO A HARMONIC FUNCTION*

BY J. L. WALSH

It is well known that if a function $u(x, y)$ is harmonic in a closed Jordan region of the (x, y)-plane, then in that closed region, $u(x, y)$ can be expanded in a uniformly convergent series of harmonic polynomials in (x, y). A more explicit result can be proved, however, and it is the object of the present note to establish such a more general theorem.

THEOREM I. *Let C be an arbitrary closed Jordan region of the (x, y)-plane, and let $w=\phi(z)$, $z=x+iy$, be a function which maps conformally the exterior of C onto the exterior of the unit circle in the w-plane so that the points at infinity correspond to each other. Let C_R denote the curve $|\phi(z)|=R$, $R>1$, that is, the transform onto the z-plane of the circle $|w|=R$.*

A necessary and sufficient condition that an arbitrary function $u(x, y)$, defined in C, be harmonic in (the closed region) C is that there should exist harmonic polynomials $p_n(x, y)$ of degree n,† $n=0, 1, 2, \cdots$, and numbers M, $R>1$, such that the inequalities

$$|u(x,y) - p_n(x,y)| \leqq \frac{M}{R^n}, \tag{1}$$

where M and R are constants, that is, independent of n and of x, y, and where $R>1$, should be valid for every point (x, y) in C.

If the polynomials $p_n(x, y)$ are given so that (1) is satisfied for every (x, y) in C, the sequence $\{p_n(x, y)\}$ converges everywhere interior to C_R and uniformly on any closed point set

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* Presented to the Society, September 9, 1926.

† A harmonic polynomial of degree n is a harmonic polynomial of the form $\sum_{p+q\leqq n} a_{pq}x^p y^q$. No assumption of the non-vanishing of the a_{pq} is made. Compare, however, the discussion given in the paper referred to in the second following footnote. A similar remark applies to the degree of $V_n(z)$.

*interior to C_R, so the function $u(x, y)$ is harmonic throughout the interior of C_R.**

If $u(x, y)$ is given harmonic in the closed region interior to C_ρ, the polynomials $p_n(x, y)$ can be chosen to satisfy (1) *with $R=\rho$, for (x, y) in C.*

Theorem I is analogous to, and will be proved by means of, the following theorem.†

THEOREM II. *Let C be an arbitrary closed Jordan region of the z-plane, and let $w=\phi(z)$ be a function which maps conformally the exterior of C onto the exterior of the unit circle in the w-plane so that the points at infinity correspond to each other. Let C denote the curve $|\phi(z)|=R$, $R>1$, that is, the transform onto the z-plane of the circle $|w|=R$.*

A necessary and sufficient condition that an arbitrary function $F(z)$ defined in C be analytic in the closed region C is that there should exist polynomials $V_n(z)$ of degree n, $n=0, 1, 2, \cdots$, and numbers M, $R>1$, such that the inequalities

$$|F(z) - V_n(z)| \leqq \frac{M}{R^n}, \tag{2}$$

where M and R are constants, that is, independent of n and of z, and where $R>1$, are valid for every z in C.

If the polynomials $V_n(z)$ are given so that (2) *is satisfied, the sequence $\{V_n(z)\}$ converges everywhere interior to C_R and uniformly on any closed point set interior to C_R, and thus $F(z)$ is analytic throughout the interior of C_R.*

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* Here and below we tacitly assume that if $u(x, y)$ is not originally supposed to be defined on the entire point set considered, then the definition in the new points is to be made by harmonic extension—or what amounts to the same thing, by means of the convergent series of harmonic polynomials. A similar remark applies in Theorem II.

† The names of S. Bernstein, M. Riesz, Faber, Fejér, Szegö should be particularly mentioned in connection with Theorem II. Proof of that theorem and detailed references to the literature are to be found in a paper by the present writer, Sitzungsberichte der Bayerischen Akademie der Wissenschaften, 1926, pp. 223–229. Compare also a forthcoming paper by Szegö in the same journal.

If $F(z)$ is given analytic in the closed region interior to C_ρ, the polynomials $V_n(z)$ can be chosen so that (1) *is satisfied for $R=\rho$ and z in C.*

If the function $u(x, y)$ of Theorem I is given harmonic in the closed region interior to C_ρ, the function

$$F(z) = u(x,y) + iv(x,y), \quad z = x + iy,$$

where $v(x, y)$ is a function conjugate to $u(x, y)$, is analytic likewise in the closed region interior to C_ρ, and hence the polynomials $V_n(z)$ exist so that inequalities (2) are satisfied for z in C and for $R=\rho$. The real part of $F(z)-V_n(z)$ is $u(x, y)-p_n(x, y)$, where $p_n(x, y)$ is a harmonic polynomial of degree n, so inequalities (1) are satisfied, $R=\rho$, (x, y) in C.

It remains merely to prove that part of Theorem I concerning itself with given polynomials $p_n(x, y)$. Here we find it convenient to prove two lemmas.

LEMMA I. *Let Γ be an arbitrary analytic Jordan curve in the (x, y)-plane and Γ' an analytic Jordan curve interior to Γ. Then there exists a constant γ depending only on Γ and Γ' such that the inequality*

$$|u'(x,y)| \leqq \mu, \tag{3}$$

where (x, y) is in or on Γ, and where $u'(x, y)$ is an arbitrary function harmonic in the closed interior of Γ, implies the inequalities

$$\left|\frac{\partial u'(x,y)}{\partial x}\right| \leqq \gamma\mu, \quad \left|\frac{\partial u'(x,y)}{\partial y}\right| \leqq \gamma\mu, \tag{4}$$

where (x, y) is in or on Γ'.

The proof of Lemma I is easy, by Green's formula:

$$u'(x,y) = \frac{1}{2\pi}\int_\Gamma u'(\xi,\eta)\frac{\partial g(\xi,\eta\,;\,x,y)}{\partial n}ds, \tag{5}$$

where g denotes the Green function for the region interior to Γ, and n the inner normal. Partial differentiation of (5)

with respect to x and with respect to y, if in the right-hand member we differentiate under the integral sign, yields inequalities (4), where γ is the larger of the upper bounds of the two quantities

$$\frac{1}{2\pi}\int_\Gamma \left|\frac{\partial^2 g(\xi,\eta\,;\,x,y)}{\partial x \partial n}\right| ds, \qquad \frac{1}{2\pi}\int_\Gamma \left|\frac{\partial^2 g(\xi,\eta\,;\,x,y)}{\partial y \partial n}\right| ds.$$

These two quantities are of course bounded when (x, y) lies (as we here suppose) on or within Γ'.

Lemma I will be used in proving the following lemma.

LEMMA II. *Let Γ be an analytic Jordan curve in the (x, y)-plane and Γ' an analytic Jordan curve interior to Γ. Then there exists a constant γ' depending only on Γ and Γ' such that the inequality* (3), $|u'(x, y)| \leqq \mu$, *where (x, y) is in or on Γ and where $u'(x, y)$ is an arbitrary function harmonic in the closed interior of Γ, implies the existence of a function $v'(x, y)$ conjugate to $u'(x, y)$ on and interior to Γ such that*

$$|v'(x,y)| \leqq \gamma'\mu, \tag{6}$$

where (x, y) is in or on Γ'.

Denote by (a, b) any fixed point interior to Γ'; then the lengths of the shortest curves in the closed interior of Γ' from (a, b) to the various points in and on Γ' have a finite upper limit L.* If we introduce the definition

$$v'(x,y) = \int_{(a,b)}^{(x,y)} \left(-\frac{\partial u'}{\partial y}dx + \frac{\partial u'}{\partial x}dy\right) = \int_{(a,b)}^{(x,y)} \frac{\partial u'}{\partial n}ds,$$

$$(x, y \text{ in or on } \Gamma),$$

where the normal n has the same relation to the direction s as the positive axis of y to the positive axis of x, the function $v'(x, y)$ is conjugate to $u'(x, y)$ in and on Γ. This integral is independent of the path of integration in Γ; we take, then, the integral over the shortest path in Γ' from (a, b) to (x, y), when (x, y) lies in or on Γ'. Inequalities (4) give us

* The reader should have no difficulty in proving this statement.

$$\left|\frac{\partial u'}{\partial n}\right| \leqq 2^{1/2}\gamma\mu,$$

so we have, if (x, y) is in or on Γ', $|v'(x, y)| \leqq 2^{1/2}\mu\gamma L$, which is of the form (6).

We return now to the proof of Theorem I. Let Γ be an analytic Jordan curve interior to C, and Γ' an analytic Jordan curve interior to Γ. If the harmonic polynomials $p_n(x, y)$ satisfying (1) are given, we have

$$|u(x,y) - p_n(x,y)| \leqq \frac{M}{R^n}, \quad |u(x,y) - p_{n-1}(x,y| \leqq \frac{M}{R^{n-1}},$$

$$|p_n(x,y) - p_{n-1}(x,y)| \leqq M\,\frac{1+R}{R^n},$$

where (x, y) is in or on Γ. There exists therefore a function $q_n(x, y) - q_{n-1}(x, y)$ defined in and on Γ and conjugate to $p_n(x, y) - p_{n-1}(x, y)$ such that we have by (6)

$$|q_n(x,y) - q_{n-1}(x,y)| \leqq \gamma' M\,\frac{1+R}{R^n},$$

where (x, y) is in or on Γ'. If we set $V_n(z) = p_n(x, y) + iq_n(x, y)$, then $V_n(z)$ is a polynomial in z of degree n and we have

$$|V_n(z) - V_{n-1}(z)| \leqq M\,\frac{(1+\gamma')(1+R)}{R^n},$$

where z is in or on Γ'. It follows that the sequence $\{V_n(z)\}$ converges in and on Γ' to a function $F(z)$, and in such a way that we have

$$\begin{aligned}|F(z) - V_n(z)| &= |[V_{n+1}(z) - V_n(z)] \\ &\quad + [V_{n+2}(z) - V_{n+1}(z)] + \cdots| \\ &\leqq M(1+\gamma')(1+R)\left[\frac{1}{R^{n+1}} + \frac{1}{R^{n+2}} + \cdots\right] \\ &= \frac{M(1+\gamma')(1+R)}{(1-R)R^{n+1}}, \qquad (z \text{ in or on } \Gamma'),\end{aligned}$$

which is of form (2). Hence by Theorem II, the sequences $\{V_n(z)\}$ and $\{p_n(x, y)\}$ converge interior to C_R', and uniformly on any closed point set interior to C_R', where C_R' is the transform onto the z-plane of the circle $|w| = R$, when the exterior of the unit circle in the w-plane is mapped onto the exterior of Γ' in the z-plane so that the points at infinity correspond to each other.

But Γ' is really an *arbitrary* analytic Jordan curve interior to C, and can be chosen as near to the boundary of C as we please. When Γ' approaches the boundary of C uniformly, it follows from the classical results of Carathéodory* on conformal mapping that the curve C_R' approaches the curve C_R uniformly, so Theorem I is established.

Theorem II is valid under a more general assumption than that stated, namely when instead of the closed interior of a Jordan curve we consider C to be an arbitrary closed limited point set not a single point whose complementary set, with respect to the entire plane, is simply connected. We cannot extend Theorem I to this most general case, but can nevertheless prove a more general result than Theorem I.

If C is an arbitrary limited region, simply or multiply connected, there may be points P, not belonging to the closed region C, which cannot be joined to the point at infinity by means of a simple arc containing no point of the closed region C. These points P make up a finite or infinite number of regions G whose boundary points are all points of the boundary of C. Thus if a sequence of harmonic polynomials or

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* Mathematische Annalen, vol. 27 (1912), pp. 107–144, especially pp. 126–127. It is convenient to think of the curve Γ' as approaching the boundary of C monotonically—that is, each successive curve Γ' lies exterior to the preceding, so that the curves C_R' likewise vary monotonically, and hence C_R' approaches C_R uniformly. For the details connected with non-monotonic approach, see Carathéodory, loc. cit.

Here we may also apply the results of Courant, Göttinger Nachrichten, 1914, pp. 101–109; 1922, pp. 69–70, or of Lebesgue, Rendiconti di Palermo, vol. 24 (1907), pp. 371–402, especially pp. 398–399. But the results of Carathéodory (his Satz V, together with the remark in §§12a and 19 on uniform convergence) are in precisely the proper form for our application.

polynomials in z converges uniformly in the closed region C, that sequence converges uniformly likewise in each of the closed regions G, for it converges uniformly on the boundary of every such region.

Such a sequence of harmonic polynomials may, however, represent different monogenic harmonic functions in C and in a region G. We give here an illustration without proof. Let C be a strip, closed at one end, that winds infinitely many times about the unit circle and approaches that circle. The circumference of the unit circle then belongs to the closed region C and the interior of that circle is a region G. The function $u(x, y) = \log (x^2+y^2)^{1/2}$ is harmonic in the closed region C and in that closed region may be represented by a uniformly convergent series of harmonic polynomials.* This series of polynomials converges uniformly on the unit circle, but interior to that circle represents not $\log (x^2+y^2)^{1/2}$, but the function which is harmonic interior to that circle and takes on the values $\log (x^2+y^2)^{1/2}$ on the circumference, namely zero.

Let us in the general case call the *extended interior of* C the closed point set $\overline{C}$ which is the totality of all points P which cannot be joined to the point at infinity by a simple arc containing no point of the closed region C. We shall then prove a theorem which is particularly interesting in view of the illustration just given.

THEOREM III. *Let C be an arbitrary limited region of the (x, y)-plane, and let B be the complementary set (with respect to the entire plane) of $\overline{C}$, the extended interior of C. Denote by $w=\phi(z)$, $z=x+iy$, a function which maps B conformally onto the exterior of the unit circle in the w-plane so that the points at infinity correspond to each other, and denote by C_R the curve $|\phi(z)|=R$, $R>1$, that is, the transform onto the z-plane of the circle $|w|=R$.*

* See a forthcoming paper by the present writer in Journal für Mathematik.

A necessary and sufficient condition that a function $u(x, y)$ defined on $\overline{C}$ (or in the closed region C) be harmonic on $\overline{C}$ is that there shall exist harmonic polynomials $p_n(x, y)$ of degree n, $n=0, 1, 2, \cdots$, and numbers M and $R>1$, such that inequalities (1) *are valid for all points (x, y) of $\overline{C}$ (or of the closed region C).*

If the polynomials $p_n(x, y)$ are given so that (1) *is satisfied for all points (x, y) on $\overline{C}$ (or all points (x, y) in the closed region C), the sequence $\{p_n(x, y)\}$ converges everywhere interior to C_R and uniformly on any closed point set interior to C_R, so the function $u(x, y)$ is harmonic everywhere interior to C_R.*

If $u(x, y)$ is given harmonic in the closed region interior to C_ρ, the polynomials $p_n(x, y)$ can be chosen to satisfy (1) *with $R=\rho$, for (x, y) on $\overline{C}$.*

If the function $u(x, y)$ is given harmonic in the closed interior of C_ρ, a conjugate function $v(x, y)$ exists which is likewise harmonic in that closed region. By the extension of Theorem II, the polynomials $V_n(z)$ can be found so that (2) is satisfied for z in $\overline{C}$ and for $R=\rho$, where $F(z)=u(x, y)+iv(x,y)$. The real part $p_n(x, y)$ of the polynomial $V_n(z)$ is such that (1) is satisfied for $R=\rho$ and for (x, y) in $\overline{C}$.

Suppose finally that the polynomials $p_n(x, y)$ are given so that (1) is satisfied for (x, y) in the closed region C. The reasoning already used is then valid, even if C is multiply connected, if a sequence Γ_n' of curves Γ' is chosen, the curve Γ_n' lying interior to C, every curve Γ_n' containing all points on and interior to Γ_{n-1}', every point interior to C likewise interior to some Γ_n'. For application of the results of Carathéodory it is sufficient to notice that no region B' containing B but different from B lies exterior to all the curves Γ_n'. Any such region B' would contain at least one interior point of either C or a region G. Such a point cannot, however, be joined to the point at infinity by a curve containing no point of the boundary of C, hence cannot be joined to the point at infinity by a curve lying wholly in the open region B'.

HARVARD UNIVERSITY

ON INDUCTIVE RELATIONS*

BY C. H. LANGFORD

1. *Introduction.* Difficulties, connected with the occurrence of reflexive fallacies, appear when we attempt to give straightforward definitions of inductive series. These difficulties arise inevitably from the fact that, if a series is to be inductive, in the ordinary sense, properties of all orders must be transmitted in the series, and it would seem that no proposition, nor any finite set of propositions, can assert that properties of all orders are transmitted.† In the discussion which follows, we shall confine attention to the definition of a particular type of inductive series, namely the order-type ω. This restriction is made for the sake of definiteness and simplicity, and it does not entail any loss of generality in the points we shall wish to illustrate.

In a note in Mind for 1923, Dr. J. E. McTaggart holds that there is a sense in which the dictum, *No proposition can be about itself*, is false;‡ and he maintains, on the contrary, that propositions in intension can be about themselves in the sense that they are applicable to themselves; and that distinctions of type are transcended in the case of these propositions. I think it just possible that this distinction is an important one, and wish to inquire what can be made of it by way of overcoming certain difficulties in the theory of types. McTaggart's view requires that we make a sharp distinction between modal and non-modal propositions, and that non-modal or material propositions be strictly subject to differences of type, in contrast to modal propositions.

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* Presented to the Society, May 7, 1927.

† Compare, however, *Principia Mathematica*, 2d ed., vol. 1, p. xliii ff. and Appendix B.

‡ *Propositions applicable to themselves*, Mind, vol. 32, p. 462. We may, for the moment, overlook the fact that this dictum itself seems to be a proposition about itself; it is a proposition in intension. Compare L. Wittgenstein, *Tractatus Logico-Philosophicus*, p. 57.

We may begin by distinguishing, roughly, three senses in which a proposition may be said to be about its subject-matter. (1) Primary propositions, propositions like "*a* gives *b* to *c*," are, in a well-understood sense, directly about the terms involved in them, and express a relation among these terms, or ascribe a characteristic to a single term. (2a) Elementary functions of primary propositions, propositions of the form $p \vee q$ for example, are not directly about the terms involved in p, q; but they are about these terms indirectly, since they restrict the truth-values of p, q,—"*at least one of the propositions p, q is true.*" Tertiary propositions, like $(p \vee q) . (r \vee s)$, are about the terms in the primary propositions still less directly; and so on. (2b) General propositions, propositions of the form $(x) . \phi x$ for example, differ from primary and elementary propositions in that proper names are replaced by apparent variables. Propositions of the form $(x).\phi x$ are not directly about the values of x; such a proposition may be rendered, Every value of ϕx is true. Propositions involving a double quantification, as for example $(\exists y):(x).\phi(x,y)$, are about the values of y indirectly, and about the values of x doubly indirectly,—some value of $(x).\phi(x,y)$, say $(x).\phi(x,b)$, is such that $(x).\phi(x,b)$ is true; and so on. Again, $\phi(x,y)$ may have elementary functions of primary propositions for values; in which case we get a combination of (2a) and (2b). Or, we may have elementary functions of functions which are not elementary, as for example $(x).\phi x. \vee .(\exists y).\psi y$. These functions are always reducible to functions which do not involve elementary functions of functions that are not elementary—the function just cited is equivalent to $(x):(\exists y).\phi x \vee \psi y$—but even when such reduction is not effected, the principle of interpretation is the same: we have a definite hierarchy of values and values of values. (3) This is the sense in which a modal proposition is about the propositions to which it applies; the sense in which, for example, "*being a proposition entails being true or false*" is applicable to a given proposition. I shall wish to express these propositions in a

somewhat different form, and will defer discussion of (3) until some other distinctions have been drawn. The contrast to be emphasized here falls between (2b) and (3), between general propositions in extension and modal propositions.

We have now to consider two senses in which an expression may be said to be a value of another expression, senses in which one expression may be validly obtained from another by substitution. (i) In a strict sense, when a proposition is a value of a function, the function characterizes the proposition; the proposition is an instance of the function. Thus, $(x).\phi x$, Every value of ϕx is true, involves the totality of instances of ϕx; and it is in connection with this sense of value that illegitimate totalities arise, when we attempt to include as values propositions which are not instances of ϕx. The same remarks apply to multiply-quantified propositions—as for example, $(x):(\exists y).\phi(x,y)$, Every value of $(\exists y).\phi(x,y)$ is true—which involve a hierarchy of values. In this sense of value, ϕa is inferred from $(x).\phi x$ by substituting a for x in ϕx; and it is essential, for substitution of this sort to be valid, that a replace x in all of the occurrences of x. This point is of some importance in connection with the occurrence of certain pseudo-propositions. For example, if we should attempt to substitute ϕx for x in ϕx, we should get $\phi(\phi x)$; but this substitution is not complete, for ϕx must be substituted for the occurrence of x in this latter expression, and so on. The regression is infinite and vicious. (ii) We have, in this case, to consider a function f' as a value of a function f in the sense that the intension of f' involves that of f, and is, in general, a further determination of the intension of f. This relation holds when f' entails f. We may begin with an illustration. If we turn to the theory of elementary functions of propositions, as given in *Principia Mathematica*, vol. 1, *1–*5, and consider any particular proposition, as for example

$$p.p \supset q. \supset .q, \tag{1}$$

there appears to be some doubt as to the precise force of

this proposition; and there are, in fact, two alternative readings, either of which might be meant. It might be meant that

(2) $$(p,q):p.p \supset q. \supset .q,$$

every value of $p.p \supset q. \supset .q$ is true. On the other hand, it might be held that (1) is a modal proposition, involving the function $p.p \supset q. \supset .q$, which could be written

(3) "$p.p \supset q. \supset .q$" cannot fail.

It is to be noted that the terms *hold* and *fail* apply to properties or functions, while *true* and *false* apply to propositions; so that (2) is about the totality of values of the function involved, while (3) asserts a modal characteristic of the function itself. Now any function which is a further determination of the function in (3), as for example

(4) $$\sim p.\sim p \supset q. \supset .q,$$

is, in a sense, a value of the function in (3). But (4) is not a value in a sense (i); for substitution of $\sim p$ for p, in sense (i), would lead to a vicious regression, since $\sim p$ would have to replace p in every occurrence of p; and in any case, the expression obtained by substitution is not a proposition, as it always is in sense (i). In the language of Mr. W. E. Johnson, (4) is a relative determinate under the function in (3) as determinable;* the function in (3) is related to (4) as "being colored" is related to "being red," and not as "x is colored" is related to "*this* is colored"—this latter being the relation of function to value in sense (i).

In the second edition of *Principia Mathematica*, the ambiguity of (1), as between (2) and (3), is decided in favor of (2).† This is natural in view of the abandonment of the real variable, as distinguished from a universal apparent variable having the whole of the asserted proposition or scope, and in view of the fact that the reference of the

* *Logic*, Part 1, p. 173.

† Introduction, vol. 1, p. xiii.

variables in (1) was confined to propositions of one order. We shall, on the other hand, adopt (3) as the interpretation of (1); and this is why numbers *1–*5 were referred to as the theory of elementary functions of propositions, rather than as the theory of elementary propositions. If we denote by p_1 a proposition of the first order, and by p_2 a proposition of the second order, then it is to be held that

$$(2_1) \qquad (p_1,q_1):p_1.p_1 \supset q_1. \supset q_1$$

follows from (3), but not conversely; and that

$$(2_2) \qquad (p_2,q_2):p_2.p_2 \supset q_2. \supset .q_2$$

follows from (3), but not conversely. Again, the proposition

$$(5) \qquad (\phi):.(\exists y):(x).\phi(x,y):\supset :(x):(\exists y).\phi(x,y)$$

follows from the modal proposition

(6) "$(\exists y):(x).\phi(x,y):\supset :(x):(\exists y).\phi(x,y)$" cannot fail;

and this holds no matter what the order of ϕ, in 5, may be. A modal proposition entails a corresponding factual universal, and it may entail many such propositions of different orders, but no factual universal entails a modal proposition.*

2. *Modal, Material, and Apparent Variables.* In distinguishing modal and material propositions, we may recognize three kinds of variables. The variables p,q in (3) and the variable ϕ in (6) are modal variables; p,q in (2) and x,y in (5) and (6) are, of course, apparent variables. Material variables occur only in properties or functions. Thus, in the function $(x):(\exists y).\phi(x,y)$, ϕ is a material variable, and must be of a definite order. In a function like "x is colored," x is a material variable; and we may, when necessary, indicate material variables, such as ϕ and x, by writing $\ddot{\phi}$ and $\ddot{x}$. Similarly, modal variables may be written $\bar{\phi}$, $\bar{x}$; ϕ, $\bar{x}$; so that (3) becomes $\bar{p}.\bar{p} \supset \bar{q}. \supset .\bar{q}$. If this analysis should

* See C. I. Lewis, *A Survey of Symbolic Logic*, Chap. V, on strict relations.

be right, it would seem that what originally gave rise to the notion of a real variable was the recognition of propositions in which what we have called modal variables occur; but that the real variable was confused, on the one hand, with the universal apparent variable, and on the other with modal variables.

3. *Modal and Material Properties.* A property or function is derived from a proposition by replacing constant constituents of the proposition by appropriate material variables; and this holds for both material and modal propositions, for the proposition is, in either case, a value in sense (i) of the derived function. Consider, for example, the material function

(7) $$\phi x \supset \phi y.$$

This function will be satisfied by any value of x which lacks ϕ, for all values of y, and by any value of y which has ϕ, for all values of x. But the modal property,

(8) "$\phi x \supset \phi y$" cannot fail,

will hold only for $x=y$, or when ϕ is some special function and x, y are so chosen that x having ϕ necessitates y having ϕ. The relation of (7) to (8) is simply that the values for which (8) holds are a proper part of the values for which (7) holds. Again, the relation of identity is not properly analyzed by the material function $(\phi).\phi x \supset \phi y$, but rather by the modal function $\phi x \supset \phi y$, that is, being a function of x necessitates being a function of y. Modal variables occur in modal propositions and in modal properties, and the occurrence of at least one modal variable is necessary for these propositions and properties; apparent variables occur in modal and material propositions and in modal and material properties; material variables occur in modal and material properties, and their presence is necessary and sufficient for a property.

4. *Definition of the Order-Type* ω. If we consider a series of the form $a_0, \cdots, a_n, \cdots$, which is inductive, what we mean by saying that the series is inductive is always strictly equivalent to what we mean by saying that, beginning with a_0, any term, a_n, can be reached step by step. We require an analysis of what we mean when we say that a_n is an inductive distance from a_0, or that a_n can be reached step by step from a_0; and it is clear that, if a proposition p_1 is to be a right analysis of a proposition p_2, then p_1 must entail and follow from p_2, that is, must be strictly equivalent to p_2. If now we consider some property f, which belongs to a_0, and is transmitted in the series, f will also belong to a_n; and we may say that it is not the case that $f(a_0)$ is true and $f(a_n)$ false. But, of course, if b_n be some term not belonging to the series, it may also be true that $f(b_n)$; so that we may say that it is not the case that $f(a_0)$ is true and $f(b_n)$ false. There is, however, this important difference: f belongs to a_n for the reason that is belongs to a_0 and is transmitted; whereas f does not belong to b_n for this reason.* Expressed otherwise, being an hereditary property belonging to a_0 entails being a property belonging to a_n; it follows from $f(a_0)$ and f is hereditary, alone, that $f(a_n)$, but not that $f(b_n)$. It might be supposed that we could express the fact that a_n is an inductive distance from a_0 by saying that a_n has every hereditary property which belongs to a_0; but this is not the case. For example, in a series of the form $a_0, \cdots, a_n, \cdots, a_\omega$, where a_ω is the limit of the progression, it might be true that a_ω has every hereditary property which belongs to a_0; whereas it is of course false that a_ω can be reached step by step from a_0, and false that $f(a_0)$ and f is hereditary entails $f(a_\omega)$.

Properties of the order-type ω will be formulated, in the usual way, on $\phi x, \psi(x, y)$, that is, on a class K and a dyadic relation R_2. There will be the usual properties for a discrete series having a first but no last element, which are material properties; and a single modal property. This modal prop-

* Compare Lewis, loc. cit., on strict implication.

erty distinguishes ω from such order-types as $\omega+{}^*\omega+\omega$, which also satisfy all of the material properties. A property is hereditary in $a_0, \cdots, a_n, \cdots$, if there is no term having this property whose immediate successor lacks it. Now t_2 immediately succeeds t_1 if, and only if,

$$\phi(t_1).\phi(t_2).\psi(t_1,t_2):.(z):\phi(z).z\neq t_2.\psi(t_1,z).\supset.\psi(t_2,z).$$

This is a function of t_1, t_2, and we may denote it by $S(t_1, t_2)$. A property f is hereditary in the series if, and only if,

$$(t_1,t_2):S(t_1,t_2).f(t_1).\supset.f(t_2)\ ;$$

and this function of f may be denoted by $T(f)$. If a_i, a_j are such that $\psi(a_i, a_j)$, then $T(f).f(a_i).\supset.f(a_j)$. But this relation is stronger, for $T(f).f(a_i)$ entails $f(a_j)$, that is,

$$T(\bar{f}).\bar{f}(a_i).\supset.\bar{f}(a_j)\ ;$$

so that the inductive property may be formulated,

p_1 $\quad (a_i,a_j):.\phi(a_i).\phi(a_j).\psi(a_i,a_j).\supset:T(\bar{f}).\bar{f}(a_i).\supset.\bar{f}(a_j).$

The following material properties are familiar:

p_2 $\quad (\exists x).\phi x;$ $\qquad$ p_3 $\quad (x).\phi x\supset\sim\psi(x,x);$

p_4 $\quad (x_1,x_2):\phi(x_1).\phi(x_2).\sim x_1=x_2.\supset.\psi(x_1,x_2)\vee\psi(x_2,x_1);$

p_5 $\quad (x_1,x_2):\phi(x_1).\phi(x_2).\sim x_1=x_2.\supset.\sim\psi(x_1,x_2)\vee\sim\psi(x_2,x_1);$

p_6 $\quad (x_1,x_2,x_3):\phi(x_1).\phi(x_2).\phi(x_3).\sim x_1=x_2.\sim x_2=x_3.$
$\qquad\qquad \sim x_1=x_3.\psi(x_1,x_2).\psi(x_2,x_3).\supset.\psi(x_1,x_3);$

p_7 $\quad (\exists x):.(y):\phi x:\phi y.\sim x=y.\supset.\psi(x,y);$

p_8 $\quad (x):::(\exists y)::.(z)::\phi x.\supset:.\phi y.\psi(x,y):.\phi z.\supset:$
$\qquad \psi(x,z).\psi(y,z).\vee.\psi(z,x).\psi(z,y).\vee.z=x\vee z=(y).$

5. *Note on Classes.* It may be suggested that modal propositions have some bearing on the customary treatment of classes through their determining functions. There are two related questions which arise in this connection; namely, whether there exist determining functions of the proper order, and whether there exist determining functions at all. We have suggested how, in certain cases, difficulties

as to order might be overcome. With regard to the second question, we wish to know that every class has a determining function; but this is not sufficient: we wish to know that every class must have a determining function. An illustration will make this clear. Suppose we have two classes α_1, α_2 and respective determining functions f_1, f_2, such that $(x).f_1(x) \supset f_2(x)$. Now we have occasion very often to make such assertions as,

(i) Every subclass of α_1 is a subclass of α_2;

and wish to express this by

(ii) $$(\phi):(x).\phi x \supset f_1(x). \supset .(x).\phi x \supset f_2(x).$$

Aside from difficulties of type, (ii) is not logically equivalent to (i) even if every subclass of α_1 has a determining function, unless it is also true that every subclass must have. But neglecting this point, let us suppose that there is some subclass β, of α_1, which has no determining function. Then, clearly, (ii) is not a right analysis of (i). But if we use a modal proposition in the analysis of (i), it seems to be unimportant whether every class has a determining function, or whether it must have; and we seem to be able to treat classes through functions whether classes have determining functions or not. For (i) may be expressed

(iii) $$(x).\bar{\phi}x \supset f_1(x). \supset .(x).\bar{\phi}x \supset f_2(x) ;$$

that is, being a function which determines a subclass of the class determined by f_1 entails being a function which determines a subclass of the class determined by f_2. Now, by hypothesis, β is a subclass of f_1 having no determining function; but (iii) is relevant to β, for it is nevertheless true that, if β had a determining function, that function would determine a subclass of f_2.

HARVARD UNIVERSITY

THREE-PARAMETER AND FOUR-PARAMETER LINEAR FAMILIES OF CONICS IN THE GALOIS FIELDS OF ORDER 2^n*

BY A. D. CAMPBELL

1. *Three-Parameter Families.* We shall denote a Galois field of order 2^n by the symbol $GF(2^n)$. We define a three-parameter linear family of conics in such a $GF(2^n)$ as the locus of all points whose coordinates x, y, z satisfy an equation of the form

$$\lambda C_1 + \mu C_2 + \nu C_3 + \rho C_4 = 0, \tag{1}$$

where

$$C_i = a_i x^2 + b_i y^2 + c_i z^2 + f_i yz + g_i zx + h_i xy = 0, \quad (i = 1,2,3,4),$$

and where the variables, coefficients, and parameters represent numbers in this domain. The conics $C_1=0, \cdots, C_4=0$ are linearly independent, and are called fundamental conics of this family. In this paper we derive the classes of these families, and we give a typical family for each class. We note that in any $GF(2^n)$ every number is a perfect square with just one square root, and $(\alpha x+\beta y+\gamma z)^2=\alpha^2x^2+\beta^2y^2+\gamma^2z^2$. We note that every such family has at least one double line.

We first divide these families into the following distinct sets.

Set I. Each family contains a net of conics reducible to the form

$$\lambda x^2 + \mu y^2 + \nu z^2 = 0. \tag{2}$$

Set II. Each family contains no net reducible to (2), but does contain a net reducible to the form

$$\lambda x^2 + \mu y^2 + 2\nu xy = 0. \tag{3}$$

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Set III. Each family contains at least two double lines, but no net reducible to (2) or (3).

Set IV. Each family contains only one double line, but has also a net reducible to the form

$$\lambda x^2 + \mu xy + \nu xz = 0. \tag{4}$$

Set V. Each family has none of the preceding properties, but has a pencil reducible to the form

$$\lambda x^2 + \mu xy = 0. \tag{5}$$

Set VI. Each family has just one double line, but none of the preceding degenerate pencils and nets.

Set I. Such a family can be put in the form

$$\lambda x^2 + \mu y^2 + \nu z^2 + \rho(fyz + gzx + hxy) = 0.$$

We easily get

$$\text{Class (1)}: \quad \lambda x^2 + \mu y^2 + \nu z^2 + \rho xy = 0. \tag{6}$$

Set II. We obtain

$$\text{Class (2)}: \quad \lambda x^2 + \mu y^2 + \nu xy + \rho(z^2 + zx) = 0. \tag{7}$$

$$\text{Class (3)}: \quad \lambda x^2 + \mu y^2 + \nu xy + \rho zx = 0. \tag{8}$$

Any transformation

$$x = \alpha_1 x' + \beta_1 y' + \gamma_1 z', \quad y = \alpha_2 x' + \beta_2 y' + \gamma_2 z', \quad z = \alpha_3 x' + \beta_3 y' + \gamma_3 z', \tag{9}$$

where $|\alpha_1, \beta_2, \gamma_3| \neq 0$, that is to send (7) into (8) must have $\gamma_1 = \gamma_2 = 0$, $\gamma_3 \neq 0$, since the net $\rho = 0$ of one family must go into that of the other. But (9) then cannot send (7) into a family lacking the term in z^2.

Set III. We can reduce such a family to the form

$$\lambda x^2 + \mu y^2 + \nu xz + \rho(cz^2 + fyz + hxy) = 0.$$

We get the following three classes, according as $cf \neq 0$; or $c = 0$, $f \neq 0$; or $c \neq 0$, $f = 0$.

(10) Class (4) : $\lambda x^2 + \mu y^2 + \nu zx + \rho(z^2 + yz) = 0.$

(11) Class (5) : $\lambda x^2 + \mu y^2 + \nu zx + \rho yz = 0.$

An argument similar to that used in the study of (7) and (8) shows that (10) and (11) are non-equivalent. Here the pencil $\nu = \rho = 0$ of (10) must go into the similar pencil of (11) by any projectivity (9).

(12) Class (6) : $\lambda x^2 + \mu y^2 + \nu zx + \rho(z^2 + xy) = 0.$

It is easy to prove (12) non-equivalent to (10) or (11).

Set IV. We put (1) in the form

$$\lambda x^2 + \mu xy + \nu xz + \rho(by^2 + cz^2 + fyz) = 0.$$

We get two classes, according as $C_4 = 0$ in the above family is a pair of real or conjugate imaginary lines.

(13) Class (7) : $\lambda x^2 + \mu xy + \nu xz + \rho yz = 0.$

(14) Class (8) : $\lambda x^2 + \mu xy + \nu xz + \rho(y^2 + z^2 + \alpha yz) = 0,$

where $C_4 = 0$ is irreducible. Any transformation (9) that is to send (14) into (13) must have $\beta_1 = \gamma_1 = 0$. But then the conic $C_4 = 0$ of (14) goes into a conic $y'^2(\beta_2^2 + \beta_3^2 + \alpha\beta_2\beta_3) + z'^2(\gamma_2^2 + \gamma_3^2 + \alpha\gamma_2\gamma_3) + \cdots = 0$. So we must have $\beta_2^2 + \beta_3^2 + \alpha\beta_2\beta_3 = 0$, $\gamma_2^2 + \gamma_3^2 + \alpha\gamma_2\gamma_3 = 0$ which give us $\beta_2 = \beta_3 = \gamma_2 = \gamma_3 = 0$ and (9) is singular.

Set V. We can put the family in one of the three following forms:

(A) $\lambda x^2 + \mu xy + \nu(z^2 + zx) + \rho(by^2 + fyz + gzx) = 0,$

(B) $\lambda x^2 + \mu xy + \nu yz + \rho(by^2 + cz^2 + gzx) = 0,$

(C) $\lambda x^2 + \mu xy + \nu(z^2 + yz) + \rho(by^2 + cz^2 + gzx) = 0.$

Case (A) gives us two classes.

(15) Class (9) : $\lambda x^2 + \mu xy + \nu(z^2 + zx) + \rho(y^2 + yz) = 0.$

(16) Class (10) : $\lambda x^2 + \mu xy + \nu(z^2 + zx) + \rho yz = 0.$

Any projectivity (9) that is to send (15) into (16) must have $\beta_1 = \gamma_1 = \gamma_2 = 0$, $\beta_2\gamma_3 \neq 0$. But then $C_3 = 0$ and $C_4 = 0$ of (10)

must go separately into conics lacking the term in y^2. This gives us $\beta_3^2=0$, $\beta_2^2+\beta_2\beta_3=0$; hence $\beta_2=0$.

Case (B) gives us two classes.

(17) Class (11) : $\lambda x^2+\mu xy+\nu yz+\rho(y^2+z^2+zx)=0.$

(18) Class (12) : $\lambda x^2+\mu xy+\nu yz+\rho(y^2+zx)=0.$

The families (17) and (18) are easily proved to be non-equivalent. They are non-equivalent to families (15) and (16) because of the presence in these two latter families of the degenerate pencil $\lambda x^2+\nu(z^2+zx)=0$.

Case (C) gives us the following class.

(19) Class (13) : $\lambda x^2+\mu xy+\nu(z^2+yz)+\rho(y^2+\alpha z^2+zx)=0.$

Set VI. We can put the family in one of two forms

(A) $\lambda x^2+\mu(y^2+xy)+\nu yz+\rho(by^2+cz^2+gzx)=0,$

(B) $\lambda x^2+\mu yz+\nu(by^2+cz^2+gzx)+\rho(b'y^2+c'z^2+h'xy)=0.$

Case (A) gives us

(20) Class (14) :

$$\lambda x^2+\mu(y^2+xy)+\nu yz+\rho(y^2+\alpha z^2+zx)=0, \quad \alpha\neq 0.$$

Case (B) gives us

(21) Class (15) :

$$\lambda x^2+\mu yz+\nu(y^2+\alpha z^2+zx)+\rho(\beta y^2+\gamma z^2+xy)=0,$$

where $\gamma=1$ or $\gamma\neq$ cube, $\gamma/\beta\neq\alpha$. The family (20) has a degenerate pencil $\lambda x^2+\mu(y^2+xy)=0$, which does not occur in (21).

2. *Four-Parameter Families.* We define a four-parameter linear family of conics in a $GF(2^n)$ by the equation

(22) $$\lambda C_1+\mu C_2+\nu C_3+\rho C_4+\sigma C_5=0,$$

where the details of notation are as indicated in the description of equation (1). It is easy to show that every such family has at least two double lines. First we assume that

(22) has a degenerate net reducible to (2). Then we assume that (22) has no net (2), but a net (3) Then we assume that (22) has neither (2) nor (3).

If (22) has a net (2) we can put the family in the form

$$\lambda x^2 + \mu y^2 + \nu z^2 + \rho xy + \sigma(fyz + gzx) = 0,$$

which gives us

(23) Class (1) : $\lambda x^2 + \mu y^2 + \nu z^2 + \rho xy + \sigma xz = 0.$

If (22) has a net (3), but not a net (2), we get

$$\lambda x^2 + \mu y^2 + \nu xy + \rho xz + \sigma(cz^2 + fyz) = 0,$$

which gives us, according as $c \neq 0$, or $c = 0$,

(24) Class (2) : $\lambda x^2 + \mu y^2 + \nu xy + \rho xz + \sigma(z^2 + yz) = 0,$

(25) Class (3) : $\lambda x^2 + \mu y^2 + \nu xy + \rho xz + \sigma yz = 0.$

If (9) is to send (24) into (25) we must have $\gamma_1 = \gamma_2 = 0$, $\gamma_3 \neq 0$. But such a transformation cannot send (24) into a family lacking the term in z^2.

If (22) has no net (2) and no net (3) we easily reduce the family to the form

(26) Class (4) : $\lambda x^2 + \mu y^2 + \nu xz + \rho yz + \sigma(z^2 + xy) = 0.$

SYRACUSE UNIVERSITY

CLARENCE ABIATHAR WALDO

BY W. H. ROEVER

It has been said that next to those who are directly laboring to extend and advance science, they contribute to its progress who prepare fitting aids to the young beginner and remove the difficulties in his way. The late Clarence Abiathar Waldo was thus outstanding as a contributor to scientific progress in this country. He was essentially a teacher, having once said of himself: "My greatest contribution to education has been my intimate contact with thousands of young men during their college period. . . . . These are my intellectual children, whose ideals I have helped to form. As a teacher of *men*, I will be best known and remembered."

He was among the pioneers in mathematical organization in this country. In 1891 he became member of the New York Mathematical Society, which was reorganized in 1894 as the American Mathematical Society. He was one of the small group of mathematicians present at the Mathematical Congress at Chicago and the Colloquium at Evanston in 1893. He was active on several committees of the Society, serving with Professors Bolza and Townsend in framing a report on the requirements for the Master's Degree (this Bulletin, vol. 10, p. 380). He was also a charter member of the Mathematical Association of America, being particularly active in the Missouri Section.

Professor Waldo's mathematical interests were chiefly in applications to engineering. He was author of a *Manual of Descriptive Geometry* (D. C. Heath, 1888), to which the remark above concerning "fitting aids to the young beginner" was particularly applicable. He followed the practice, now too much neglected, of having the student make models of the space objects under consideration in the course. His address as retiring chairman of Section D (now M) of the American Association for the Advancement of Science on the subject *The relation of mathematics to engineering* (Proceedings, vol. 53 (1904)) is interesting and instructive. In the Association for the Advancement of Science he was also secretary of the council, 1903–04, and general secretary and editor of the Proceedings, 1904–05. He was very active in the Society for the Promotion of Engineering Education, serving as treasurer, 1900, and editor of its Proceedings, 1901–03. The Indiana Academy of Science was also fortunate in having Professor Waldo as editor of its Proceedings, 1896–98, and as president, 1898. In connection with his work in the Academy, Professor Waldo told with amusement of a bill which had been introduced in the Indiana Legislature fixing a "new and correct value of π" as a definite rational number. The bill was being favorably considered when he heard of it, and quick action was necessary to prevent its passage. It was due to his efforts that the bill was rejected. He was a member of Sigma Xi and Phi Beta Kappa.

Other activities than those strictly scientific also claimed his interest, and attention. He was a member and president, 1890, of the Indiana

College Association. His interest in athletics was very keen. He was arbitrator of the Chicago Conference of Colleges, 1901–12, and of the Ohio Conference of Colleges, 1903–05.

Professor Waldo received the degrees A.B. and A.M. from Wesleyan University in 1875 and 1878, respectively, and the degree Ph.D. from Syracuse University in 1894. He spent the academic year 1882–83 in study at the universities of Leipzig and Munich.

He held the following academic positions: Instructor in Mathematics and Registrar, Wesleyan University, Conn., 1877–81; Professor of Mathematics, 1883–91, and Acting President, 1885–86, 1888–89, Rose Polytechnic Institute; Professor of Mathematics, De Pauw University, 1891–95; Head Professor of Mathematics, 1895–98, Purdue University; Professor of Mathematics, 1908–10, Thayer Professor of Mathematics and Applied Mechanics and Head of the Department of Mathematics, 1910–17, Professor Emeritus after 1917, Washington University. After leaving Washington University, he worked actively on exemption boards in New York City in 1917–18. In 1919 he was College Visitor for the Carnegie Foundation.

Professor Waldo was born in Hammond, New York, January 21, 1852. He died in Bridgeport, Conn., October 1, 1926. He married Miss Abby Wright Allen, August 2, 1881, in Stamford, Conn. He had two children, Alice Goddard and Clarence A. (deceased).

His pleasing personality and his kindly nature endeared him to all who knew him. His high ideals were built on deep religious foundations and were reflected in his daily life.

WASHINGTON UNIVERSITY

EDDINGTON ON CONSTITUTION OF STARS

The Internal Constitution of the Stars. By A. S. Eddington. Cambridge University Press, 1926. viii+407 pp.

The author says that this book was written between May 1924 and November 1925 and that during this period theoretical papers in stellar constitution in the Monthly Notices alone amounted to more than 400 pages. He has tried to cover everything up to November 1925 and, in proof, some further material up to March 1926. Composition under such circumstances is difficult even for one who has contributed very actively to the subject for the previous decade, and particularly when he has been currently engaged in a running fight with his most eminent compatriot specializing in the same field. A popular account of the subject may be found in Science (May 6, 1927, pp. 431–438) by Menzel, and there is no need here to repeat it. Suffice it to remark that this book reads like a romance to those who are not deterred by the mathematics whether because they can follow it or because they ignore it; the difficulties of composition do not show in the style.

One may raise the question: Is it romance or science? We all recall the "Cosmic Crucibles" of G. E. Hale. We see only the thinnest fumes hovering over the crucibles; how much do we know about the conditions within them? Certainly very little. The highest steady temperatures with which we can work in the laboratory are only a few thousand degrees; by spectral observations on the exterior of stars we record spectra which by the law of Wien (or Planck) we estimate at most as toward a very few tens of thousands; just what the temperatures are, or even whether there is anything truly corresponding to temperature in the explosive discharges of Anderson we perhaps do not know. When therefore through a course of reasoning we set 40,000,000 degrees as the normal temperature of the center of a sun we are stretching our laboratory laws a great deal on the side of extrapolation toward infinity, we are dealing certainly not with scientific facts, in the sense of verifiable facts of observation, we are wholly in the realm of theory, in that of faith which has ever had a higher psychic appeal than fact.

True, we work in our vacuum tubes at high potentials with high speed electrons and observe their effects upon matter; true, too, that to produce free electrons of such speeds under conditions of thermal equilibrium would require intense temperatures; but how safe is the passage from an observed high tension electric dis-equilibrium with its multifarious manifestations to an inferential intense thermal equilibrium? And then there are the tremendous pressures; what are their effects on the imagined and inaccessible phenomena we are studying? Bridgman's highest laboratory pressures do not take us much if any below our lithosphere and yet they at least approach those that justify our speaking of the compressibility of the molecules and atoms. May it be that in the upper stellar atmospheres

the luminous effects are chiefly electrical, as in our vacuum tubes or possibly in our aurora, rather than thermal as such, and that in the stellar interiors the physical state is one governed more by pressure than by temperature?

The author does not minimize the physical difficulties. What he does is to take our finite physical laws of observation as they are, add to them some inferential generalizations, assume both to hold everywhere even under conditions far more extreme than any realized in the laboratory, and upon this to build a mathematical physical theory of the internal constitution of the stars, which in turn he tries to justify, and with a large measure of success, upon the record of astronomical, chiefly astrophysical, classification of the stars. This is a perfectly sound procedure. The business of a scientific theory is to bind together the scientific observations. We have certain variables or parameters $a, b, c, \cdots$ which represent the facts observed; we desire to find rational relations between them; we introduce certain auxiliary unknown variables $x, y, z, \cdots$ with laws connecting them to $a, b, c, \cdots$ and among themselves; we ultimately eliminate the unknowns and come out with the desired relations between the knowns and verify those relations on the facts. In a field so actively worked as stellar physics is today, both observationally and theoretically, in a field in part so much in dispute as is this between Eddington and Jeans, changes may well come; such is science in the romantic, perhaps better in the heroic or epic phase when an Achilles fights his Hector for some lady who, forsooth, belongs to a third (or fourth) party.

There is something more involved than the rational correlation of the observed quantities $a, b, c, \cdots$, some constraints on the manner of that correlation; the real riddle is the permanence of this planet. Geology and paleontology need half a billion years, radioactive studies set the age of the earth's crust as around a billion years, we apparently must think of the luminous life of a star as measured in centuries of billions of years. Where does the energy come from? Evolution goes too slowly for the physics we know and can only be expounded on extensive extrapolations. Laplace, Helmholtz, Kelvin, all who have tried to set a time scale by old fashioned physics have fallen far short of the mark; our present hope seems to reside in the commutation of material mass into radiant energy through Einstein's equation $E=mc^2$, each disappearing gram of matter giving rise to 9×10^{20} ergs of radiation. The synthesis of hydrogen into helium, when, if, and as it takes place, would by this equation release a store of energy vast in comparison with ordinary chemical reactions, but apparently a bit too small in amount to maintain for its life the radiation of a star; we apparently have to fall back on some sort of super-radioactivity (Jeans) or on the coalescence of electron and proton into nothingness (Eddington).

After the immediate satisfaction of our needs for individual sustenance and for the continuation of the race, our most imperious demand seems to be to solve the infinite problems of our finite minds—a truly imperial ambition. It is right and proper that we should offer a crown to such a Caesar as our author. At the risk of marking ourselves as some miserable

Cinna we may make a few additional comments quoting from p. 163: "At the beginning of 1924 the giant and dwarf theory of Hertzsprung and Russell was almost universally accepted. . . . . I do not think it is too blunt an expression to say that this is now overthrown; at least it has been gutted, and it remains to be seen whether the empty shell is still standing." Note the great change in two years. Things are moving fast, not faster perhaps than a genius may develop new theory, but surely far faster than new facts of observation may be revealed and established. And herein lies a source of possible future overthrow for any present theory; we need more and better facts and they are slow to get. Moreover we need facts that are not colored or selected by our preconceived but not yet established theories. There is a marked tendency for an accepted, albeit perhaps not established, theory to bring to light those facts which may perchance support it and to obscure those which might contravert it. This is particularly true when the argument is statistical as is now common in astronomy. I do not say that the author is unfair; the question is one of judgment and his judgment should be as good as that of anybody and certainly incomparably better than mine; I merely cite his inevitable liabilities. In speaking of the synthesis of hydrogen into helium he says (p. 301): "We do not argue with the critic who urges that the stars are not hot enough for this process; we tell him to go and find *a hotter place*." I am not finding fault with the facts that we have but merely urging that we need better ones and more of them.

Although the chief interest of the author is in the deep interiors of the stars and in astronomical observations his text is full of developments of and suggestions for physics and of astronomical conditions on the surface of or exterior to the stars. A list of the chapter headings gives the best short summary of the content of the book. I. Survey of the Problem. II. Thermodynamics of Radiation. III. Quantum Theory. IV. Polytropic Gas Spheres. V. Radiative Equilibrium. VI. Solution of the Equations. VII. The Mass-Luminosity Relation. VIII. Variable Stars. IX. The Coefficient of Opacity. X. Ionization, Diffusion, Rotation. XI. The Source of Stellar Energy. XII. The Outside of a Star. XIII. Diffuse Matter in Space. There is a table of physical and astronomical constants, a long list of references with some brief summaries of their content, and a good index. If any reader thinks the author at times is too zealous, too dogmatic, let him persevere until he can distinguish brilliancy of style from fixation of mind, even unto these words from the final paragraph: "The history of scientific progress teaches us to keep an open mind. I do not think we need to feel greatly concerned as to whether these rude attempts to explore the interior of a star have brought us to anything like the final truth. We have learned something of the varied interests involved."

EDWIN B. WILSON

CULLIS ON MATRICES

Matrices and Determinoids. By C. E. Cullis. University of Calcutta Readership Lectures. Cambridge University Press, Vol. III, Part 1, 1925. xviii+681 pp.

The first two volumes of this treatise were reviewed in this Bulletin (vol. 26 (1920), pp. 224–233). The present volume continues the development, which has become more extensive than at first contemplated, so that the third volume has had to appear in two parts. The second part, it is stated, will deal chiefly with structural matrices. The author's avowed aim is to carry forward the program of Sylvester, who conceived a magnificent treatise on universal algebra, which he never carried out. The present treatise to a large extent is realizing this program. To Professor Cullis, matrices means practically all of algebra, and it is easy to see from his standpoint that every expression in algebra, whether it depends upon one or upon more variables, is in reality a function of some sort of matrix. To the reviewer this is tantamount to saying that every function of one or more variables is ipso facto a function of a vector of a space of the same number of dimensions, or of various vector products in such space. The difference is simply one of point of view or mode of statement. It is simpler, in the reviewer's opinion, to make the whole subject a branch of the theory of hypernumbers, than to build up a theory of "sets," although this is really following the tradition of Sir W. R. Hamilton, the creator of quaternions. The author insists rightly upon the matrix as a single entity, and considers that a *pure* calculus will be determined by a system of scalar numbers, so that he would introduce hypernumbers later, the ordinary quaternion, for instance, being a product of a scalar matrix and a matrix whose elements are the usual $1, i, j, k$. While this is logical, and is a justifiable line of development, it seems to the reviewer to complicate the subject unnecessarily. The author further considers that on the applied side, every physical entity can be represented by a matrix, and he certainly would find plenty of justification for this view in modern physics. His assertion (p. vi) that the expressions would be two-dimensional tables, hardly seems tenable, however. For witness the modern tensor theory.

The scope of this volume is best indicated by a brief indication of the contents of the various chapters. The first three (XX, XXI, XXII) deal with rational integral functions of any number of variables, irresoluble and irreducible functions and factors, common factors, resultants, eliminants, discriminants, and common roots. Symmetric functions are considered specially in Chapter XXII. In Chapter XXIII begins the consideration of rational integral functional matrices, that is, matrices whose elements are rational integral functions of scalar variables, $x, y, z, \cdots$. The definition is given and properties proved of irreducible and irresoluble divisors of such matrices. These divisors are rational integral functions of the same variables as the matrix, and are of order i when they will divide every minor determinant of order i which arises

from the matrix. A *regular* minor determinant of order i is one which, with respect to an irreducible or irresoluble divisor t, does not vanish identically, and is not divisible by a higher power of t than that power which is the highest that will divide all minors of the order i. The terms invariant factors, and elementary divisors, of the classic literature, are replaced in this chapter by *potent divisor*. The greatest common divisors D are called maximum factors, the invariant factors E are called maximum divisors, the divisors of these are called potent factors and potent divisors. These are all rational in the domain considered. The former elementary divisors might have been irrational. The potent factors and divisors of the product of two matrices are considered. Following this the same divisors are considered for *compartite* matrices, that is, matrices which may be considered to be the *direct sum* (in the hypernumber sense) of simpler matrices. The next chapter, XXIV, takes up the treatment of *equipotent* matrices, that is, matrices which are *equivalent* in the sense of having the same rank, and the same potent divisors. The equipotent transformations are discussed in detail. With regard to these topics the reviewer will remark only that the fundamental elements of a matrix are what he has previously called its *shear regions*. When these are given everything is virtually given and it would be simpler to treat the subject on such a basis. The two chapters just discussed introduce the idea of rationality for a given domain, which is after all an *arithmetic* notion as contrasted with an algebraic notion. The reviewer has in mind that algebra must deal with what he calls *hypernumbers*, which are frequently called in France *relative numbers*, that is to say, numbers with qualitative signs. Even the introduction of $+$ and $-$ was necessary to produce *algebra*. Mere literal notations do not take one out of the region of *arithmetic*. In Arithmetic we find the root of the notion of *domain of rationality*. Its extension to algebraic fields followed. Thus in all discussions of such topics as we have before us we find these two aspects, the hypernumber structure, and the domain of rationality, or sometimes of integrity, as for instance in Dickson's Arithmetic of Algebras.

Chapter XXV is concerned with rational integral functions of a square matrix. It introduces the notions of *latent roots*, and the *characteristic matrix*, which is the matrix $\phi - \lambda I$, where I is the identity matrix, and the elements of the matrix ϕ are constant. It includes also the determination of all *rational integral equations* satisfied by ϕ. At the end of the chapter is Frobenius' solution of the matrix equation $\psi^2 = \phi$. The Hamilton-Cayley equation is developed.

Chapter XXVI develops the subject of *equimutant transformations*, or transformations of the form $A(\ )A^{-1}$ where A is undegenerate. This transformation corresponds directly to what is usually called a transformation in the theory of groups, or in the theory of linear homogeneous groups to an orthogonal transformation. It might be added it is also in the general theory of vectors in N dimensions, a rotation. It enables us to bring a matrix with constant elements to the canonical form. Also the idea of *transmute* is introduced. By this is meant that if $\phi = \alpha\beta$ where α and β are of rank r, the first with m rows and r columns, the second with

r rows and m columns, then $\phi_1=\beta\alpha$ is the transmute of ϕ, a notion connecting with previous results (vol. I, page 154) and useful in stating the theorem that two square matrices with constant coefficients are connected by an equimutant transformation when and only when they have a common first transmute. Transmutes are not unique, there being an infinity of first transmutes for the matrix ϕ, which, however, are all transformable into one another by equimutant transformations. In fact, the best way to arrive at a first transmute is to consider the matrix as a linear vector operator which annuls a region of dimensions equal to the nullity, and if for the region left κ is the identity matrix, then a first transmute of ϕ is $\kappa\phi\kappa$. Or in the array form, if we transform the matrix so that it has zero columns equal in number to the nullity, then by erasing the same number of rows at the bottom, producing a square matrix, we will have a first transmute. There is much material in this chapter which is related to the characteristic equation of a matrix, including the standard theorems.

Chapter XXVII deals with *commutants*. A commutant X is a matrix solution of the equation $AX=XB$ where A and B are square matrices. It is either a zero matrix or has a determinate structure. If A and B have only constant elements X is non-zero only if A and B have a latent root in common. When A and B are canonical square matrices with the same latent root, X is a general "ruled compound slope." The ideas advanced in connection with this notion have a direct connection with what the reviewer has previously called "*associative units*" (Transactions of this Society, vol. 4 (1903), 251–287). These play an important part in the linear associative algebras. There is a great deal of new material in the chapter which can scarcely be detailed here. It evidently is in preparation for the developments of the second part of volume three. Chapter XXVIII deals with commutants of commutants, which are commutants of every commutant of a given matrix A. They are proved to be the rational integral functions of A. Chapter XXIX deals with invariant transformands which are solutions of the equation $AXB=X$. When there are solutions which are non-zero they have a definite structure. It is evident that if B is invertible the invariant transformands go back into the commutant list.

The theory of matrices, in whatever notation it may be expressed, has a growing importance. From the study of finite matrices have followed many recent developments that involve the properties of infinite matrices. The study of quadratic forms on an infinity of variables, either denumerable or non-denumerable, the various developments of integral equations and of linear operators in general, and other related subjects, all lead back to the fundamental structure of matrices. These in turn are special cases of hypernumbers, and consequently we find that in hypernumbers we have as universal an algebra as we may at present study. The matrices are practically the associative hypernumbers, and special structures in these forms give the special algebras. The non-associative systems however are very much more numerous and very much more complicated. There has been up to the present time little use for such systems, perhaps due to the fact that their algebra has not been worked out, but this will be done in

the future. It is evident that from the study of such systems associativity, important as it is, is of trifling importance in the general problems of structure. It may be then that the day will come when the matrix will be of only very limited importance in the study of structural physics, and the non-associative hypernumbers will give us the keys to the universe.

It is nevertheless a matter for congratulation to Professor Cullis that he is carrying out such a heavy piece of work as the present development of matrices, and it is to be hoped that he may bring it to a successful conclusion.

J. B. SHAW

STENSTRÖM ON THE 27 LINES

Synthetische Untersuchungen des Systems von 27 Geraden einer Fläche dritter Ordnung. By Olof Stenström. Upsala, Appelbergs Boktryckeri Aktiebolag, 1925. 128 pp.

It was a matter of course, in the history of geometry, that the cubic surface should be much studied. The zest with which many have pursued this study was not a matter of course but was largely due to the elegant configuration of the 27 lines on the surface. Since their fortunate discovery by Cayley and Salmon in 1849, they have been essential to nearly all investigations concerning it—investigations in which conspicuous names are Schläfli, Clebsch, Sylvester, Cremona, Sturm, Klein, Reye, Zeuthen, Schur, Henderson, and Baker. No decade has, indeed, been without notable work on the surface of third order. Since 1920 the contributions of two men deserve especial mention. The first was E. Stenfors,* who worked on the projective transformations of a Schläfli double-six into itself, as well as on like transformations of the whole system of lines, and their groups. Olof Stenström followed in 1925 with the booklet which is the subject of this review.

If, contrary to custom, a formal Table of Contents is included in a review, it may perhaps be justified by the fact that the book itself contains none, although such a table would aid both those who wish to read it and those who wish to learn more quickly of its content.

CHAPTER		PAGE
	Introduction—Historical	1
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* *Die Schläflische Konfiguration von zwölf Geraden einer Fläche dritter Ordnung*, 1921. *Über die Geradenkonfiguration einer Fläche dritter Ordnung, bzw. Klasse*, 1922. Both memoirs are contained in Series A, vol. 18, of that valuable periodical which the Union List of Serials names Suomalaisen Tiedeakatemian Toimituksia, but which the Revue Semestrielle designates—equally correctly—as Annales Academiae Scientiarum Fennicae. One who searches for a volume of this rather scarce journal wishes that there were agreement on nomenclature.

CHAPTER		PAGE
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III.	Partial systems of lines (The heading is omitted, but should follow line 2.)	23
IV.	Correspondence of the configuration to the division of the 6 indices in two halves	26
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IX.	Singularity of trilinearity determining the surface; regions of surfaces with singular systems of lines	58
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From the title two things are clear: that Stenström follows the synthetic method which had its first great impetus in the study of the cubic surface with Cremona's memoir of 1868*; and that his researches concern the lines primarily, not the surface. Nevertheless, the Table of Contents indicates, among its wealth of subjects, three chapters (V, VI, and IX) in which the surface is treated also.

Any one of the 27 lines meets ten others in pairs of an involution. In the case of twelve lines, which happen to constitute a double-six, the involution is elliptic; on the other lines it is hyperbolic. Here we are considering, of course, the surface whose lines are real and distinct. This fact, early recognized by Klein and Zeuthen, is the basis for Stenström's detailed analysis of the various figures formed by the lines; as a result the geometry, always alluringly intricate, becomes truly elaborate. Thus our author distinguishes three types of planes determined by lines of the system, four types of double-sixes, twelve of skew triples of lines, five of Steinerian triedral pairs, and so on. The dissection of the surface by means of each kind of double-six is displayed; in particular, it appears that all parabolic points lie within those ten triangles whose boundaries carry hyperbolic involutions.

The last chapter, devoted to the cross ratios of quadruples of planes through lines of the system, continues work begun by Kasner,† but brings

* *Mémoire de géométrie pure sur les surfaces du troisième ordre*, Journal für Mathematik, vol. 68.

† *The double-six configuration connected with the cubic surface and a related group of Cremona transformations*, American Journal of Mathematics, vol. 25 (1903).

results in a new and decidedly elegant form. Let us, for instance, consider the five planes of Type II—the type in which each of the three lines has a hyperbolic involution, with the intersecting lines outside the plane giving two pairs of points on each segment bounded by lines in the plane. Let the other four planes through each of the three lines have cross ratio λ_n; this will be the same for the three lines, but will differ from plane to plane. Then these five cross ratios satisfy the symmetric equation

$$(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5)(1-\frac{1}{\lambda_1}-\frac{1}{\lambda_2}-\frac{1}{\lambda_3}-\frac{1}{\lambda_4}-\frac{1}{\lambda_5})+1$$
$$-\lambda_1-\lambda_2-\lambda_3-\lambda_4-\lambda_5=0.$$

A notable characteristic of Stenström's work is his ingenuity in notation, and in the use of diagrams. The capitalization of the usual a_n and b_n, in order to set off that double-six whose involutions are elliptic, was fairly obvious. Decidedly clever is the indication of degenerate double-sixes of Type IV, by means of the vertices and centers of two regular pentagons—in each pentagon six points for the six indices. This type of double-six

$$\begin{array}{cccccc} A_i & C_{np} & A_k & C_{mp} & A_l & C_{mn} \\ C_{kl} & B_m & C_{il} & B_n & C_{ik} & B_p \end{array}$$

obviously individuated by the indices i, k, l, "is represented by (one of) the ten triangles with an obtuse angle which can be formed from these points." The other three indices give an acute triangle in the other pentagon. 26 cases of degeneracy, caused by coincidence of conjugate lines in such double-sixes, are enumerated; and the triangles drawn in the pentagonal skeleton indicate at once the collapsed double-sixes, the coincident lines, the number and multiplicity of the lines and planes.

The pentagonal representation does duty only so long as the lines are real; admission of imaginary lines raises the number of types to 75. These 75 cubic surfaces are spread out on a family tree. From the common ancestor —the system of 27 real and distinct lines—one passes downward and to the right whenever lines come to coincidence, then upward, but still to the right, when they separate again and become imaginary. Thus the level on which the symbol for a particular system (or surface) is found shows at once the number of collapsed double-sixes.

All diagrams are decidedly neat, even if they lack the perfection of Henderson's plates* and the gaiety of Stenfors' black, red, and green pictures.

Proof-reading on Stenström's book was faulty. However, most of the errors were corrected in the copy supplied the reviewer, and it is to be hoped that these corrections will become part of the standard equipment of the book. One of them, indeed, is distressing (page 97, line 23), annulling a predicate and allowing the subject to survive as best it may.

In the excellent historical introduction we are puzzled by a reference to a "Probeartikel der Encyclopädie der Mathematischen Wissenschaften

* *The Twenty-Seven Lines upon the Cubic Surface*, Cambridge Tracts, No. 13.

von 1896"; for the publication of the encyclopedia began in 1898, the article on cubic surfaces has not yet appeared, and Stenström's "Literaturverzeichnis" gives no item dated 1896. There is confusion (page 22) in the discussion of Schur quadrics associated with various types of double-sixes. "Eine *nicht nullteilige* Fläche zweiter Ordnung teilt den Raum in zwei Bereiche. . . . . Liegen (die Geraden des Paares) auf derselben Seite (der Fläche), so ist sie *nullteilig*."

The configuration which Stenström (page 26) names "Sturmsche Doppelfünfseite" is not that which Sturm himself describes.* A double-five satisfying Sturm's specifications, but not Stenström's, is

$$\begin{matrix} C_{23} & C_{45} & C_{26} & C_{34} & C_{56} \\ C_{46} & C_{36} & C_{35} & C_{25} & C_{24}. \end{matrix}$$

The reverse is true of

$$\begin{matrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56}. \end{matrix}$$

On page 37 it is stated that "einander nicht schneidende Gerade bleiben einander nicht schneidende oder fallen ganz und gar zusammen." Yet when, in

$$\begin{matrix} A_1 & A_2 & A_3 & C_{56} & C_{64} & C_{45} \\ C_{23} & C_{31} & C_{12} & B_4 & B_5 & B_6, \end{matrix}$$

corresponding pairs come to coincidence, all lines become concurrent, and A_1 and A_2 intersect without coinciding. This section could be stated more carefully. And what is the meaning of the assertion (page 63) that a double-six whose lines have become imaginary is "im ganzen betrachtet immer noch reell"?

Stenström devotes some attention to the Sylvester pentahedron, whose vertices are the double points of the cubic surface's Hessian; and he hints that in this matter he plans further investigation, which is to develop the relation between the system of lines and the pentahedron. It is to be hoped that we shall learn of the success of this attempt; for the present book establishes its author's ability very substantially to extend our knowledge in this field of geometry.

E. S. ALLEN

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* *Die 27 Geraden der cubischen Fläche*, Mathematische Annalen, vol. 23 (1884), p. 296.

SHORTER NOTICES

A Course of Differential Geometry. By J. E. Campbell. Oxford, Oxford University Press, 1926. xv+261 pp.

The manuscript for this book was completed by Professor Campbell shortly before his death in 1924. It was prepared for the press by Professor E. B. Elliott. The book was evidently planned to prepare the reader to understand differential geometry as applied in modern relativity theory. The methods of tensor analysis have been employed throughout.

The first chapter gives a brief but quite satisfactory introduction to the tensor theory associated with a symmetric quadratic differential form with n independent variables. The Christoffel symbols of three and four indices are defined and discussed and the operation of covariant differentiation is introduced without being so called. At the end of this chapter an Einstein space is defined as one in which certain tensor components vanish but no explanation of the definition is given at this point.

The next nine chapters contain a treatment of differential geometry of ordinary space, tensor methods being used almost exclusively. Particular attention is given to the following subjects: equivalence of forms, geodesics, curvature deformation of surfaces, congruences, curves in space and on a surface, ruled surfaces, minimal surfaces, conformal representation, orthogonal surfaces.

In order to understand these nine chapters the reader must have previously acquired the main facts of differential geometry. Little explanation outside of the analysis is offered and the geometrical concepts are usually introduced without definition. The purposes of this part of the book undoubtedly are to show the power of vector methods and to prepare the way for extensions to hyperspace.

In the last four chapters we return to the consideration of the quadratic form with n independent variables and are thus led to geometry in n-way space. As a generalization of the straight line in the plane we have the geodesic. Gauss's measure of curvature has as a natural extension Riemann's measure of curvature, in which a tangential n-fold takes the place of the tangent plane.

The fundamental form for an $(n+1)$-way space may be taken as

$$\phi du^2 + b_{ik} dx_i\, dx_k \qquad (i, k = 1, \cdots, n),$$

where ϕ, b_{ik} are functions of $x_1, \cdots, x_n$ and u. The surface $u=0$ is an arbitrary surface in the space. This $(n+1)$-way space is an Einstein space if for all such surfaces lying in it

$$\sum \frac{1}{R_i R_k} + \frac{1}{2} A = 0,$$

where the first term represents the sum of the products, taken two at a time, of the reciprocals of the principal radii of curvature of the surface, and where

A is a certain invariant dependent on the Christoffel symbols of four indices for the n-way space of the surface. For $n=2$ the above formula reduces immediately to Gauss's well known formula for the product of the reciprocals of the principal radii of curvature of a surface. The above definition is easily seen to be equivalent to that stated in the first chapter.

A given n-way space is always surrounded by an Einstein $(n+1)$-way space whose generation can be accomplished by infinitesimal methods. The Einstein space which surrounds the n-way space $u=0$ is stationary if ϕ and b_{ik} are independent of u.

The last two chapters of the book are given to the discussion of n-way space of constant Riemannian curvature and of n-way space as a locus in $(n+1)$-way space.

Professor Campbell planned to add an appendix treating of the relation of the contents of the book to the physics of Einstein. Such an appendix would undoubtedly have added greatly to the understanding of the physical meanings of some of the mathematical results.

There is a complete absence of references in the book. Important results are not stated in the form of theorems or made to stand out on the page in any way.

E. B. STOUFFER

Les Fondements des Mathématiques. By F. Gonseth. Paris, Blanchard, 1926.

This book, which commences with a preface by J. Hadamard, deals with the foundations of geometry and kinematics and the relation between mathematics and logic.

In the first few chapters a discussion is given of various sets of axioms for euclidean space, the axioms of Geiger being compared with those of Hilbert.

Following a chapter on the continuum there is next a short chapter on the compatibility and independence of the axioms of a system. The author then passes on to the construction of continua in which he makes use of Jordan curves, topological transformations and groups of movements. His program is expressed as follows: Topology should be constructed with the aid only of the axioms of order and continuity.

Chapter VI is devoted to non-euclidean geometry and the next chapter to the relation between theory and experience. The discussion of time and relativity is enlivened by a short dialogue in which Bergson, Metz, LeRoux, Fabre, and the author are supposed to take part.

There is next a chapter on the idea of motion and general relativity and the book closes with a chapter on mathematics and logic in which special attention is devoted to the vicious circle of Weyl, Brouwer's remarks on the principle of the excluded third, Weyl's formulation of the continuum and Hilbert's logical foundations of mathematics. There is also a short article on formal logic.

H. BATEMAN

Oeuvres complètes de Christiaan Huygens publiées par la Société Hollandaise des Sciences. Tome Quinzième. Observations Astronomiques. Système de Saturne. Travaux Astronomiques, 1658–1666. La Haye, Martinus Nijhoff, 1925. Pp. 622.

This fifteenth volume of the works of Huygens is a model of what can be done for the convenience of readers. A general table of contents (sommaire) containing twelve entries is supplemented by a fuller one (pièces et mémoires) covering eight pages. There are alphabetical indexes to persons and to subject matter. Numerous footnotes contain explanations and cross references. The longer documents appear in the original Latin and in French translation. Facsimile reproductions of parts of manuscripts and original drawings add to the interest of the book. Introductions to the various parts contain information in great detail. Among the mathematical notations of interest are the Cartesian sign of equality (the sign for taurus turned so that the opening is on the left) and the "scratch method" used in extracting square roots.

Huygens' early astronomical observations relate mainly to Saturn, Jupiter and their companions. Huygens and his brother Constantine constructed their own telescopes. The first one, 12 feet in length, promptly revealed a new satellite of Saturn. This instrument and a 23 foot telescope furnished data to solve the enigma relating to the varying appearance of Saturn. Galileo had seen it threefold. Hevelius saw it in three parts. Fontana made drawings widely distorted. Huygens in 1655 drew Saturn as a sphere with two handles or arms on opposite sides and extending in opposite directions. When Saturn was near the sun, in January, 1656, the arms disappeared. Nevertheless, Huygens had a hypothesis which he published in 1656, disguised as an anagram: aaaaaaaccccdeeeeghiiiiiiillll mmnnnnnnnnnoooppqrrsttttuuuuu. Later in 1656, the handles reappeared and gradually changed so as to confirm his theory. In 1657 and 1658, the rings became somewhat plainer, and in 1659 he published his *Systema Saturnium* in which he explained his anagram as signifying a Latin sentence which is in translation: "It is surrounded by a slender flat ring, everywhere distant from the surface, and inclined to the ecliptic."

FLORIAN CAJORI

Einführung in die Analytische Geometrie. By A. Schoenflies. Berlin, Julius Springer, 1925. x+304 pp.

This book is volume XXI of the series *Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen*, and in general excellence of treatment is in keeping with the other volumes of the series. The author states in the preface that there is perhaps a greater contrast between the methods of elementary and advanced analytic geometry than those of any other branch of mathematics, and this book will be very helpful to the student who is in this transition stage in his study of geometry.

For the purpose of this review the book will be divided arbitrarily into four parts including respectively Chapters I–V, VI–VIII, IX–XII, and XIII–XVII inclusive; the first two of these parts being approximately

50 pages each and the last two 80 pages each. To this is added an appendix and index of 40 pages.

The first five chapters include the fundamental elementary topics which are studied in a first course in analytic geometry. These are developed in the familiar non-homogeneous analytical language but from a broader and more general point of view than the method of the ordinary text book. After explaining various coordinate systems the equations of the conics are derived in both the cartesian and polar systems together with the parametric equations of the circle and ellipse. The distance of a point from a line is found by means of projections. This relation and the transformation of coordinates precede the chapter on the straight line in which the importance of the Hesse normal form is shown and which includes a good treatment of the theory of two lines and of three lines.

In Chapters VI–VIII inclusive the fundamental framework of modern analytic geometry is constructed. The projective method predominates as this gives the most natural approach to homogeneous coordinates. The author develops this part in logical order and with great clearness. It includes line coordinates, duality, the complete 4-point and 4-line, the theorems of Desargues (perspective triangles) and Pascal, double ratio, and the theory of harmonic pairs. Linear substitution is shown to be a projective relation; quadratic involutions are discussed; and the conic is obtained as a locus or envelope of corresponding projective forms. Homogeneous coordinates are developed first from cartesian point and line coordinates, then linear projective coordinates are introduced. Finally the general plane homogeneous coordinates are defined by means of a triangle of reference, and the ratios of these coordinates are shown to be given by means of a double ratio as soon as the unit point is fixed.

Chapters IX–XII inclusive deal principally with a more extensive treatment of the conics, the general equation of the second degree, and collineations in the plane. Orthogonal pencils of circles, circular points and minimal lines, and transformation by reciprocal radii are included in the chapter on the circle. In the following chapters are discussed confocal conics, conjugate diameters, invariants, polar systems and involutions, the pencil of conics, collinear and reciprocal transformations, and the theorems of Pascal and Brianchon.

The last five chapters are devoted to the geometry of three dimensions following a procedure analogous to that used in the plane. Coordinate systems, dual elements and loci, collineations in space, a discussion of the quadric surfaces both with non-homogeneous and with the general homogeneous coordinates, and the polar system are the principal subjects considered. Useful related algebraic topics are given a supplementary treatment in the appendix, including determinants, linear equations, and transformations. The book ends with a selected list of 115 exercises and problems.

After introducing the homogeneous coordinate systems it seems unfortunate that the author did not make more extensive use of them. Had the equation of the circle, for instance, been given with an inscribed triangle of reference it would have opened to the student a most attractive

method for investigating the geometry of the circle and triangle. However, the arrangement of the material is logical and successive topics are developed in a masterly fashion. It should be kept in mind that the author has attempted to fulfill not only the formal requirements of the Wissenschaften but also the demands of the student in geometry both as to the selection and arrangement of material. In this sense the book is a compromise; to quote from the preface "ein einführendes Lehrbuch soll in erster Linie ein Lernbuch sein." A few minor errors in type were observed but these do not detract from the merits of the book. It is a welcome addition to our libraries and will be a valuable reference for students in their first course in modern analytic geometry.

J. I. TRACEY

La Logique des Mathématiques. By Stanislas Zaremba. Mémorial des Sciences Mathématiques, No. 15. Paris, Gauthier-Villars, 1926. 52 pp.

In this number of the Mémorial des Sciences Mathématiques, Professor Zaremba, of the University of Cracow, views logic as a "theory of deductive demonstration" and sketches a "logistic" by means of which a "complete" mathematical demonstration can be effected. In the course of his exposition, in which he "carefully refrains from engaging in psychologic and philosophic speculations," the author formulates fundamental problems in the logic of mathematics still awaiting solution, and gives an explanation of the paradoxes in aggregate theory based on a distinction between the notions *class* and *category*. The text is followed by a good-sized bibliography.

The author excludes from his logic the logic of classes and the logic of relations—theories that "really should not enter into the domain of a general theory of demonstration." His logic is a logic of propositions consisting (1) of a theory of "propositional identities" and (2) of a theory of propositions of logic of the "second kind." The first of these seems to be the theory of "elementary" propositions of Whitehead and Russell's *Principia Mathematica*, whose "theory of deduction" is "the most complete exposition of the theory of propositional identities." The second theory is presumably that of "apparent variables" of the *Principia*. The author, however, does not accept Russel's "theory of types," which has brought into logic "numerous obscurities and extreme complications."

B. A. BERNSTEIN

Esquisse d'Ensemble de la Nomographie. (Mémorial des Sciences Mathématiques, No. 4.) By Maurice d'Ocagne. Paris, Gauthier-Villars, 1925. 68 pp.

This is one of the little volumes on interesting mathematical subjects now being published under the direction of Professor Henri Villat as Mémorial des Sciences Mathématiques. It covers in four chapters of twenty-eight sections most of what is known of nomographic theory. To the advanced student the treatment offers interest and charm for the presentation has authority, brevity and elegance. The engineer or physi-

cist seeking help in the actual construction of nomograms will be disappointed, for there are no applications. The reader recognizes at page 39 that the problem of determining under what conditions the equation in three variables $f(z_1,z_2,z_3)=0$ shall have the determinant form

$$\Delta_{123} = |f_{ij}g_ih_i| = 0, \qquad (i = 1, 2, 3),$$

is fundamental and is referred to the literature for the solution. The generalized equation in six variables with corresponding nomograms consisting of three curve nets is also briefly considered. Chapter IV is devoted to nomograms with movable inscribed elements and affords suggestions for the study of nomograms depending on geometric invariants under displacements in the plane. There is a useful bibilography of thirty sources and the printing is almost without error. On page 49 there should be a factor 3 before the parenthesis in the fifth line from the bottom. On page 39, j and k should be deleted from the parenthesis line 14 from the bottom, and v should replace ν in equations 3), 4), and 5), page 30.

L. I. HEWES

Théorie Générale des Séries de Dirichlet. (Mémorial des Sciences Mathématiques, No. 17.) By G. Valiron. Paris, Gauthier-Villars, 1926. 56 pp.

The aim of the series to which this work belongs is to present in brief and compact form the most important results and outlines of methods in various mathematical subjects of current interest. This particular number on Dirichlet series gives an excellent survey of this interesting field which is now growing so rapidly.

One naturally thinks of comparing this book with the only other work devoted exclusively to this subject, namely *The General Theory of Dirichlet's Series* by Hardy and Riesz (Cambridge Tracts in Mathematics and Mathematical Physics, No. 18, published in 1915). They are somewhat similar in general plan and style. That there was a real need for a new book on the subject is seen from the fact that of the bibliography of nearly 150 references given by Valiron, over forty per cent were published since 1915 when the Hardy and Riesz work was issued.

A curious mistake is made on pages 31–32, in a brief discussion of the Hölder and Cesàro methods of summation of divergent series, where the names of these two methods are interchanged.

To those already interested in the subject and to those who wish know something of the results and methods without devoting too much time to it, this compact summary should prove very valuable.

L. L. SMAIL

NOTES

The April, 1927, number of the Annals of Mathematics ((2), vol. 28, No. 2) contains the following papers: *Some relations between a continuous curve and its subsets*, by H. M. Gehman; *Transformations of one principal equation into another*, by R. Garver; *On related maxima and minima*, by N. Miller; *On Fredholm's integral equations, whose kernels are analytic in a parameter*, by J. D. Tamarkin; *On the existence of the absolute minimum in space problems of the calculus of variations*, by L. M. Graves; *The mixed mean function*, by C. A. Shook; *The focal point for the problem of Lagrange with one variable end point*, by L. H. McFarlan; *Differential invariants of affinely connected manifolds*, by T. Y. Thomas and A. D. Michal; *Linear congruences in a general arithmetic*, by H. L. Olson; *Note on a differential equation*, by J. M. Thomas; *On the class of a covariant tensor*, by J. W. Alexander.

The Deutsche Mathematiker-Vereinigung, the Gesellschaft für angewandte Mathematik und Mechanik, the Deutsche Physikalische Gesellschaft, and the Gesellschaft für Technische Physik will meet together at Kissingen in September, 1927.

The International Geodetic and Geophysical Union will meet at Prague, September 3–10, 1927. Dr. William Bowie and Mr. W. D. Lambert, of the Coast and Geodetic Survey, have been appointed as delegates of the United States by the president of the National Academy of Sciences and the chairman of the National Research Council.

Professor Edmund B. Wilson has been elected president of the American Academy of Arts and Sciences.

The Reale Accademia dei Lincei has conferred its royal prize in mathematics on Professor Leonida Tonelli of the University of Bologna. One of the prizes founded by the (Italian) Ministry of Public Instruction has been awarded to Professor Enea Bortolotti, of the Reale Liceo Artistico di Bologna.

The Belgian Accademy announces the following subject for its prize in mathematics in 1928: *A contribution to the theory of quasi-analytic functions.* Competing memoirs should reach the Academy by August 1, 1928.

Dr. L. Prandtl, professor of applied mathematics at the University of Göttingen, has been awarded the gold medal of the Royal Aeronautical Society, in recognition of his work in aerodynamics.

The Franklin Institute has awarded Franklin Medals and certificates of honorary membership to Professor Max Planck, of the University of Berlin, and to Dr. G. E. Hale, honorary director of the Mount Wilson Observatory.

The University of Münster has conferred an honorary doctorate on Johann Georg Hagen, director of the Vatican Observatory, on the occasion of his eightieth birthday.

Professors A. S. Eddington, of Cambridge University, and G. H. Hardy, of Oxford University, have been elected corresponding members of the Bavarian Academy of Sciences.

Professors A. Einstein, A. A. Michelson, G. Mittag-Leffler, and W. Nernst, and Madame Curie have been elected honorary members of the Russian Academy of Sciences.

Awards from the Heckscher Fund at Cornell University have been made to Professor J. I. Hutchinson, for the salary of an assistant to spend part of his time on the study of properties of functions defined by certain Dirichlet series, and to Professor Arthur Ranum, for an assistant to spend part of his time on the study of the principles of duality in the differential geometry of surfaces and curves.

Dartmouth College has conferred honorary doctorates on President Max Mason, of the University of Chicago, Professor F. P. Brackett, of Pomona College, and Professor D. C. Miller, of the Case School of Applied Science.

The University of Wisconsin has conferred the honorary degree of doctor of science on Professor G. D. Birkhoff, of Harvard University.

Princeton University has conferred the honorary degree of doctor of science on Professor A. A. Michelson, of the University of Chicago.

Professor E. H. Moore, of the University of Chicago, has received the honorary degree of doctor of science from the University of Kansas and from Northwestern University.

President G. D. Olds, of Amherst College, has received an honorary doctorate of laws from Wesleyan University.

Brown University and the University of Rochester have conferred honorary doctorates on Professor M. I. Pupin, of Columbia University.

Yale University has conferred the honorary degree of doctor of science on Professor A. N. Whitehead, of Harvard University.

The University of Michigan has conferred the degree of doctor of science on Professor emeritus Alexander Ziwet, of that University.

Professor Svante Arrhenius, director of the Nobel Institute for Physical Chemistry, has retired.

Dr. Wolfgang Krull, of the University of Freiburg i. Br., has been promoted to an associate professorship of mathematics.

Dr. Josef Lense, of the University of Vienna, has been appointed associate professor of applied mathematics at the Munich Technical School.

Professor Arnold Sommerfeld, of the University of Munich, has been called to the chair of theoretical physics at the University of Berlin, as successor to Professor Max Planck, retired.

Professor A. Lafay, of the Ecole Polytechnique, has been appointed maître de conférences in mathematics at the Institut National Agro nomique.

Professor H. Villat, of the University of Strasbourg, has been appointed to the newly established chair of the mechanics of fluids at the Sorbonne.

Mr. A. S. Besicovitch, of the University of Leningrad, has been appointed university lecturer in mathematics at Cambridge University for a term of three years.

A fellowship in mathematics of $1,000 annually has recently been established at Brown University by Mr. H. D. Sharpe.

Assistant Professor L. C. Bagby, of the University of South Dakota, has been appointed professor of mathematics at the Linsly Institute of Technology.

Professor L. S. Dederick, of St. Stephen's College, has been appointed mathematician at the Aberdeen Proving Ground.

Assistant Professor W. W. Elliott has been promoted to a professorship of mathematics at Duke University.

Dr. P. A. Fraleigh has been appointed assistant professor of mathematics at the University of Vermont.

Assistant Professor G. I. Gavett has been promoted to an associate professorship of mathematics at the University of Washington.

Assistant Professor Howard Justice has been appointed associate professor at the University of Cincinnati.

Associate Professors J. H. Kindle and Edward S. Smith have been promoted to professorships of mathematics at the University of Cincinnati.

Professor P. H. Linehan, of the College of the City of New York, has been made Director of the Evening Session and of the Division of Vocational Studies.

Miss Muriel Metz has been promoted to an assistant professorship of mathematics at the University of Cincinnati.

Mr. T. H. Milne has been appointed assistant professor of mathematics at the University of Manitoba.

Dr. H. H. Neelley, of Yale University, has been appointed associate professor of mathematics at the Carnegie Institute of Technology.

Dr. C. A. Nelson, of Johns Hopkins University, has accepted a position at Rutgers University.

Dr. B. P. Reinsch, of the University of Illinois, has been appointed associate professor of mathematics at the Southern Methodist University.

Assistant Professor E. E. Whitford, of the College of the City of New York, has been promoted to an associate professorship of mathematics.

Dr. H. A. Zinszer, of the Mississippi State College for Women, has been appointed professor of mathematics at Hanover College.

The following appointments as instructor in mathematics are announced:

Columbia University, Dr. R. G. Archibald (associate), Mr. Joseph Feld, Dr. B. O. Koopman, Mr. H. W. Raudenbush, Dr. P. A. Smith, Mr. G. C. Vedova;

Kenyon College, Mr. Benedict Williams;

Northwestern University, Mr. H. L. Garabedian;

Yale University, Mr. T. C. Benton.

Dr. Wilhelm Ahrens, author of books on mathematical recreations, died at Rostock, April 23, 1927.

Dr. Felix Auerbach, professor of theoretical physics at the University of Jena, is dead.

Dr. Ludwig Burmester, formerly professor of mathematics at the Munich Technical School, is dead.

Dr. Viktor Eberhard, associate professor of mathematics at the University of Halle, is dead.

Dr. Ludwig Maurer, formerly professor of mathematics at the University of Tübingen, died January 10, 1927.

Dr. F. Thieme died March 9, 1926, at the age of seventy-three.

Dr. Anton Gmeiner, professor of mathematics at the University of Innsbruck, died January 11, 1927.

Dr. Ludwig Grossmann, of Vienna, is dead.

Dr. Alois Walter, professor of mathematics at the Montanistische Hochschule at Leoben, is dead.

Professor Giuseppe Bagnera, of the University of Rome, died May 13, 1927.

Mr. G. L. Cathcart, fellow of Trinity College, Dublin, and editor of Salmon's mathematical works, died March 26, 1927.

Professor W. G. Bullard, of Syracuse University, died February 16, 1927. Professor Bullard had been a member of the American Mathematical Society since 1899.

Professor W. W. Johnson, of the United States Naval Academy, died May 14, 1927, at the age of eighty-three. Professor Johnson had been a member of the American Mathematical Society since 1891.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

BETSCH (C.). Fiktionen in der Mathematik. Stuttgart, Fr. Frommans Verlag, 1926. 24+372 pp.

BISWAS (R. C.). The theory of equations and the complex variable. Calcutta, Chuckervertty, Chatterjee and Company, 1926. 8+269 pp.

BOWMAN (F.). Elementary algebra. Part II. New York and London, Longmans, 1927. 431 pp.

BRODETSKY (S.). Sir Isaac Newton. A brief account of his life and work. London, Methuen, 1927. 12+161 pp.

BROWN (J. T.) and MARTIN (A.). The elements of plane trigonometry. London, Harrap, 1927. 200 pp.

BURKAMP (W.). Begriff und Beziehung. Leipzig, Meiner, 1927.

CHARLIER (C. V. L.). Naturwetenskapens matematiska principer av Isaac Newton. Lund, Gleerups Förlag, 1927.

CZAIKOWSKYJ (N.). Algebra. Part II. (In Ukrainian.) Prague, 1926.

DICKSON (L. E.). Algebren und ihre Zahlentheorie. Mit einem Kapitel über Idealtheorie von A. Speiser. Zurich, Füssli, 1927. 307 pp.

DOMBROWSKI (M. C. M.). Canon geometricus oder explizite definitorische Grundlagen der Geometrie. 2te neubearbeitete Auflage. Minsk, Verlag "Kultura," 1926. 8 pp.

DURELL (C. V.). Mathematical tables (four figures). London, Bell, 1927. 40 pp.

ENGEL (F.). See LIE (S.).

FLADT (K.). Gewöhnliche Differentialgleichungen. (Mathematische Physikalische Bibliothek.) Leipzig, Teubner, 1927. 67 pp.

FORDER (H. G.). The foundations of euclidean geometry. Cambridge, University Press, 1927. 12+349 pp.

GHEURY DE BRAY (R.). See REYMOND (A.).

GIBBS (R. W. M.) and RICHARDS (G. E.). Mathematical tables. 2d edition, revised. London, Christophers, 1926. 16 pp.

JANET (M.). Les systèmes d'équations aux dérivées partielles. (Mémorial des Sciences Mathématiques, No. 21.) Paris, Gauthier-Villars, 1927.

KEYSER (C. J.). Thinking about thinking. New York, Dutton, 1926. 91 pp.

KLEPPISCH (K.). Willkür oder mathematische Überlegung beim Bau der Cheops pyramide? München, Oldenbourg, 1927. 38 pp.

LANDAU (E.). Vorlesungen über Zahlentheorie. 3 Bände. 12+360+8+308 +8+341 pp.

LIE (S.). Gesammelte Abhandlungen. 6ter Band: Abhandlungen über die Theorie der Transformationsgruppen, 2te Abteilung. Anmerkungen zum sechsten Band. Herausgegeben von F. Engel. Leipzig, Teubner, und Oslo, Aschehong, 1927. 24+752+186 pp.

LODGE (O.). Modern scientific ideas, especially the idea of discontinuity. London, Benn, 1927. 79 pp.

LOSSKIJ (N. O.). Handbuch der Logik. Autorisierte Übersetzung nach der 2te, verbesserte und vermehrte Auflage, von W. Sesemann. Berlin, Teubner, 1927.

MARTIN (A.). See BROWN (J. T.).

MULLINS (G. W.) and SMITH (D. E.). Freshman mathematics. Boston, Ginn, 1927. 6+386 pp.

PASCAL (E.). Repertorium der höheren Mathematik. 2te, völlig umgearbeitete Auflage herausgegeben von E. Salkowski und H. E. Timerding. 1ter Band: Analysis. 2ter Teilband. Leipzig, Teubner, 1927. 12+494 pp.

PATTERSON (C. H.). Problems in logic. New York, Macmillan, 1926. 12+331 pp.

REYMOND (A.). History of the sciences in Greco-Roman antiquity. Translated by R. Gheury de Bray. London, Methuen, 1927. 10+245 pp.

RICHARDS (G. E.). See GIBBS (R. W. M.).

ROTHSTOCK (W.). See RUSSELL (B.).

RUSSELL (B.). Unser Wissen von der Aussenwelt. Uebersetzt von W. Rothstock. Leipzig, Meiner, 1926. 331 pp.

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WENNER (F.). Praktische Rechenbildkunde (Nomographie). Aachen, Aachener Verlag, 1926. 6+78 pp.

WHITEHEAD (A. N.) and RUSSELL (B.). Principia mathematica. 2d edition. Volume 3. Cambridge, University Press, 1927. 8+491 pp.

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ANGLAS (J.). D'Euclide à Einstein. Relativité et connaissance. Paris, Librairie Stock, 1926. 121 pp.

AUERBACH (F.). Die Grundbegriffe der modernen Naturlehre. Leipzig, Teubner, 1926. 128 pp.

BENNETT (T. A.). See YOUNGSON (P.).

BISCONCINI (G.). Esercizi e complementi di meccanica razionale. Milano, Libreria Editrice Politecnica, 1927. 480 pp.

BOEHM (F.). Versicherungsmathematik. II: Lebensversicherungsmathematik. Einführung in die technischen Grundlagen der Sozialversicherung. (Sammlung Göschen.) Berlin, de Gruyter, 1926.

BOHLE (H.). Electrical photometry and illumination. A treatise on light and its distribution. 2d edition, revised and enlarged. London, Griffin, 1925. 16+427 pp.

BOUASSE (H.). Emission, chaleur solaire, éclairage, théorie de l'émission. Paris, Delagrave, 1926. 500 pp.

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BOUTARIC (A.). Thermodynamique et chimie, d'après la 2e édition de l'ouvrage de P. Duhem. Paris, Hermann, 1926. 6+650 pp.

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Much needless expense and many errors can be avoided. The editors of several mathematical journals have agreed upon the following suggestions.

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3. Keep a copy, and send the editors two copies, if you can.
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5. Underline theorems with blue pencil (avoid ink).
6. Follow our recent styles in abbreviations, footnotes, etc.
7. Write carefully the (often misunderstood) capitals C K P S V W X Z.
8. Write **e**, not ε. Write very carefully $\gamma\ \eta\ \kappa\ \lambda\ \nu\ \tau\ \upsilon\ \chi\ \omega$.
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11. Use the solidus (/) to avoid fractions in solid lines.
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13. Use extra symbols to avoid complicated exponents.
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16. Some samples of unusual types available on monotype machines follow. A more complete list of all such types will be sent on request.

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★ Light Greek Superiors—$^{\Delta}$ and $^{\alpha\ \beta\ \gamma\ \cdots}$ (all except ι and o).

★ Light Greek Inferiors—$_{\Delta\Lambda\Sigma\Omega}$ and $_{\alpha\ \beta\ \gamma\ \cdots}$ (all except ι and o).

* Boldface Greek—$\boldsymbol{\alpha\ \beta\ \delta\ \varepsilon\ \zeta\ \eta\ \theta\ \mu\ \nu\ \xi\ \pi\ \rho\ \sigma\ \omega}$ and $\boldsymbol{\Omega}$.

* Lightface German—$\mathfrak{a\ b\ c\ d\ p\ q}\ \mathfrak{A\ B\ C\ D\ E\ F\ G\ H\ J\ L\ M\ P\ Q\ R\ S\ T\ U\ V\ W\ X\ Y\ Z}$.

* Boldface German—$\mathfrak{b}\ \mathfrak{A\ B\ D}$

Script (special font $\mathcal{A\ B\ C} \cdots$ (all). No lower case manufactured.

* Hebrew— ℵ א ב ב troublesome to handle.

★ Dashed Italics—$\overline{A}\ \overline{a}\ \overline{B}\ \overline{b}\ \overline{C}\ \overline{c}\ \overline{E}\ \overline{e}\ \overline{F}\ \overline{f}\ \overline{G}\ \overline{g}\ \overline{H}\ \overline{I}\ \overline{\imath}\ \overline{K}\ \overline{k}\ \overline{M}\ \overline{m}\ \overline{N}\ \overline{n}\ \overline{P}\ \overline{p}\ \overline{q}\ \overline{R}\ \overline{r}\ \overline{s}\ \overline{t}\ \overline{u}\ \overline{v}\ \overline{X}\ \overline{x}\ \overline{Y}\ \overline{y}\ \overline{Z}\ \overline{z}$

★ Tilda Italics—$\tilde{A}\ \tilde{a}\ \tilde{e}\ \tilde{N}\ \tilde{n}\ \tilde{O}\ \tilde{r}\ \tilde{u}\ \tilde{y}$

★ Tilda Greek—$\tilde{\alpha}\ \tilde{\epsilon}\ \tilde{\eta}\ \tilde{\omega}\ \tilde{\varpi}$

★ Dashed Greek—$\bar{\alpha}\ \bar{\beta}\ \bar{\gamma}\ \bar{\delta}\ \bar{\eta}\ \bar{\theta}\ \bar{\mu}\ \bar{\nu}\ \bar{\rho}\ \bar{\omega}\ \bar{\Gamma}$

★ Dotted Italic—$\dot{a}\ \ddot{a}\ \dot{e}\ \ddot{e}\ \ddot{g}\ \ddot{\imath}\ \dot{m}\ \dot{n}\ \dot{q}\ \dot{r}\ \ddot{r}\ \dot{u}\ \ddot{u}\ \dot{x}\ \ddot{x}\ \dot{y}\ \ddot{y}\ \dot{z}\ \ddot{z}$.

★ Dotted Greek—$\dot{\eta}\ \ddot{\eta}\ \dot{\theta}\ \ddot{\theta}\ \dot{\xi}\ \dot{\psi}\ \ddot{\psi}\ \ddot{\omega}$ (single dotted $\zeta\ \phi\ \delta\ \beta\ \gamma$; double dotted γ readily available).

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* Additional characters readily available at small cost.

★ Matrices for additional characters are made upon special orders and necessitate a delay of from four to eight weeks and average expense of $4.50 per matrix.

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

NEW YORK CITY, October 29, 1927.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than October 8. At the request of the Program Committee, Professor F. D. Murnaghan will give an address on *Modern hydrodynamical theory, with special reference to aerodynamics.*

BERKELEY, CALIFORNIA, October 29, 1927. (The San Francisco Section.)

Abstracts must be in the hands of the Secretary of the Section, Professor B. A. Bernstein, Berkeley, California, not later than October 8. At the request of the Program Committee, Professor E. R. Hedrick will give an address on *Extensions of the theory of functions of a complex variable.*

ST. LOUIS, MISSOURI, November 26, 1927. (The Southwestern Section.)

Abstracts must be in the hands of the Secretary of the Section, Professor E. B. Stouffer, Lawrence, Kansas, not later than November 5.

NASHVILLE, TENNESSEE, ANNUAL MEETING, December 27-28, 1927.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than November 29. At the usual joint meeting with other organizations, Professor James Pierpont will represent this Society, and will give an address on *Mathematical rigor, past and present.* The fifth Josiah Willard Gibbs lecture, under the auspices of this Society and the American Association for the Advancement of Science, will be delivered by Professor E. W. Brown.

R. G. D. RICHARDSON,
Secretary of the Society.

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

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WHOLE NUMBER 350

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